Solutions Massachusetts Olympiad Level I 2010

1. $\dfrac{70\text{ miles}}{1\text{ hour}} \cdot \dfrac{1\text{ hour}}{60\text{ minutes}} \cdot \dfrac{1\text{ minute}}{60\text{ seconds}} = \dfrac{7\text{ miles}}{360\text{ seconds}} \rightarrow \dfrac{360\text{ seconds}}{\text{mile}} = 51\tfrac{3}{7} \approx 51.43$

2. Let x be the side of the square, so its perimeter is $4x$. Then the dimensions of the rectangle are $\dfrac{x}{3}$ by x giving a perimeter of $2x + \dfrac{2x}{3} = \dfrac{8x}{3}$. Set $\dfrac{8x}{3} = 24 \rightarrow x = 9$ so $4x = 36$.

3. Let x equal the number of shots the goalkeeper stopped before the big game and let y equal the total number of shots on goal before the big game. Then $\dfrac{x}{y} = \dfrac{4}{5}$ and $\dfrac{x}{y+15} = \dfrac{1}{2}$ giving $x = \dfrac{4y}{5}$ and $2x = y + 15$. Solving, we obtain $x = 20$.

4. There are 100 teeth and 99 gaps so the area equals $10 \cdot 199 - 99 = 1891$.

5. $\dfrac{8}{11} = .727272...$ Remove the first 2 obtaining .772727 . . . Set $N = .772727...$ making $100N = 77.2727...$ Subtract N from $100N$ obtaining $99N = 76.5 \rightarrow N = \dfrac{765}{990} = \dfrac{17}{22}$.

6. Consider the following table where t is the total time of the trip and d is the total distance:

Driver	Rate	Time	Distance
Father	r_1	$\dfrac{4t}{5}$	$\dfrac{3d}{5}$
Son	r_2	$\dfrac{t}{5}$	$\dfrac{2d}{5}$

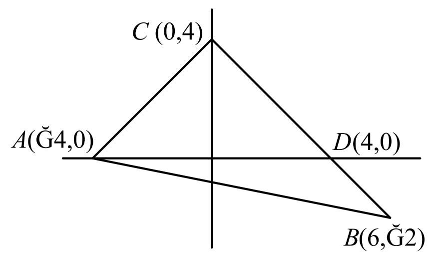
Then $\dfrac{r_1 \cdot \frac{4t}{5}}{r_2 \cdot \frac{t}{5}} = \dfrac{\frac{3d}{5}}{\frac{2d}{5}} \rightarrow \dfrac{4r_1}{r_2} = \dfrac{3}{2} \rightarrow \dfrac{r_1}{r_2} = \dfrac{3}{8}$.

7. If Al is telling the truth, then Betty, Carl, and Debby are lying, meaning that Carl did not do it, Debby did not do it, and again Debby did not do it. We have a situation where just one person is telling the truth. This means that Betty did it. On the other hand, if Betty is telling the truth, then Carl did it, but Al is lying, so Al did it as well, a contradiction. If Carl is telling the truth, then both Carl and Debby are telling the truth and that can't be. If Debby is telling the truth, then both Carl and Debby are telling the truth and that can't be. The answer is Betty (B).

8. The expression is clearly negative. Set it equal to *x*. If we square it we obtain $x^2 = \frac{5}{4} - \sqrt{\frac{3}{2}} - 2\sqrt{\frac{25}{16} - \frac{3}{2}} + \frac{5}{4} + \sqrt{\frac{3}{2}} = \frac{5}{2} - 2\sqrt{\frac{25}{16} - \frac{24}{16}} = \frac{5}{2} - 2 \cdot \frac{1}{4} = 2$. We choose the negative solution to $x^2 = 2$.

9. From $AB + BO = 10 - OC$ we obtain $AB^2 + 2AB \cdot BO + BO^2 = 100 - 20 \cdot OC + OC^2$. Since $AB^2 + BO^2 = OC^2$ and $AB \cdot BO = 5$, we have $2 \cdot 5 = 100 - 20 \cdot OC \rightarrow OC = \frac{9}{2}$.

10. For $x \geq 0$, $|x| + y = 4 \rightarrow y = 4 - x$ and for $x < 0$, we have $y = 4 + x$. The two graphs are shown to the right as bold-faced lines. The other graph intersects the graph of $|x| + y = 4$ at $A(-4,0)$ and $B(6,-2)$. Using $\overline{AD}$ as the base of each triangle, the area of the region equals the area of ΔACD plus the area of ΔADB, giving $\frac{1}{2} \cdot 8 \cdot 4 + \frac{1}{2} \cdot 8 \cdot 2 = 16 + 8 = 24$.



11. $x^2 + bx + (b+1) = 0 \rightarrow x = \frac{-b \pm \sqrt{b^2 - 4 \cdot 1 \cdot (b+1)}}{2}$. The difference in the roots is $\sqrt{b^2 - 4b - 4}$. Thus, $\sqrt{b^2 - 4b - 4} - (b+1) = 1$, making $b = -1$. Then $c = 0$ and $d = 1$ and the equation was $x^2 - x = 0$. The sum is 0.

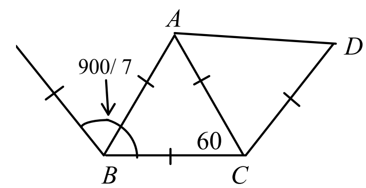
12. Subtracting the second from the first gives $3y = 2k^2 - k$, so that *y* is positive whenever $k(2k-1) > 0 \rightarrow k < 0$ or $k > \frac{1}{2}$. Multiply the second equation by 4 and subtract the first from the second obtaining $3x = 4k - 2k^2$, so *x* is positive whenever $2k(2-k) > 0 \rightarrow 0 < k < 2$. Thus, both *x* and *y* are positive if $\frac{1}{2} < k < 2$.

13. $\frac{1}{\log_2 N} \cdot \frac{1}{\log_N 8} \cdot \frac{1}{\log_{32} N} \cdot \frac{1}{\log_N 128} \rightarrow \frac{\ln 2}{\ln N} \cdot \frac{\ln N}{\ln 8} \cdot \frac{\ln 32}{\ln N} \cdot \frac{\ln N}{\ln 128} =$

$\frac{(\ln 2)(5 \ln 2)}{(3 \ln 2)(7 \ln 2)} = \frac{5}{21}$.

14. $(x + yi)(1 + 2i) = 4 + 2i \rightarrow x + yi = \frac{4 + 2i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} \rightarrow x + yi = \frac{4 - 8i + 2i + 4}{1^2 + 2^2} \rightarrow$ $x + yi = \frac{8 - 6i}{5} \rightarrow x = \frac{8}{5}, y = -\frac{6}{5}$. Thus, $x + y = \frac{2}{5}$.

15. The measure of an interior angle of a regular 7-gon is $\frac{(7-2)\cdot 180}{7} = \frac{900}{7}$. Then $m\angle ACD = \frac{900}{7} - 60 = \frac{480}{7}$.



Since $AC = CD$, $m\angle ADC = \frac{180 - \frac{480}{7}}{2} = \frac{390}{7}$.

16. We can consider three cases. If 5 is at the end, there are $\frac{6!}{2!2!} = 180$ possible strings. If the string ends in both 2's, there are $\frac{5!}{2!} = 60$ possible strings. If the string ends in one 2, then there are 5 places to put the second 2 that are not next to the last place 2 and $\frac{5!}{2!}$ distinct ways to arrange the other letters, giving $\frac{5 \cdot 5!}{2!} = 300$ possible strings. Thus the answer is $180 + 60 + 300 = 540$.

17. $x^3 - 1 = k(x-1) \rightarrow (x-1)(x^2 + x + 1) = k(x-1)$. Thus, $x = 1$ or $x^2 + x + (1 - k) = 0$ and we want the quadratic to have a double root. Then $1 - 4(1-k) = 0 \rightarrow 4k - 3 = 0$, making $k = \frac{3}{4}$.

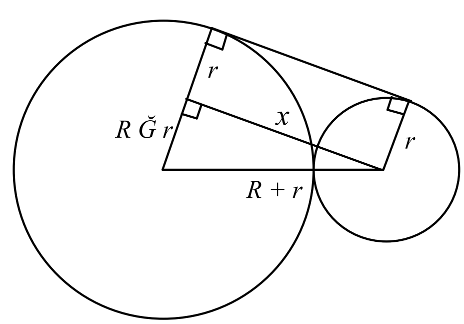
18. The palindrome can be written as $ABCBA$ where $A \neq 0$ and the letters don't have to represent distinct digits. Since the number is divisible by 11, we know that the sum of the digits in the odd positions minus the sum of the digits in the even positions is a multiple of 11, namely 0, –11, or 11 in this case.

Starting with $2A + C - 2B = 0$, we let $A = 1$, making $C = 2B - 2$. With $B = 1$ we obtain 11011, with $B = 2$ we obtain 12221, with $B = 3$ we obtain 13431, and with $B = 4$ we obtain 14641. Turning to $2A + C - 2B = -11$, again let $A = 1$, obtaining $C - 2B = -13$. Here $C \neq 0$ or 2, but if $C = 1$, $B = 7$, giving 17171, if $C = 3$, $B = 8$, giving 18381, if $C = 5$, $B = 9$, giving 19591. Turning to $2A + C - 2B = 11$, let $A = 1$, giving $C - 2B = 9$. If $B = 0$, then $C = 9$ and we obtain 10901. Thus, the three smallest palindromes are 10901, 11011, and 12221. The sum of their digits is 23.

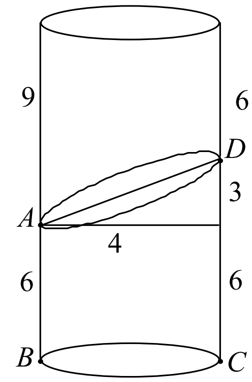
19. $\cot x + \cot y = b \rightarrow \dfrac{1}{\tan x} + \dfrac{1}{\tan y} = b \rightarrow \dfrac{\tan x + \tan y}{\tan x \cdot \tan y} = b \rightarrow \dfrac{a}{\tan x \cdot \tan y} = b$. Thus, $\tan x \cdot \tan y = \dfrac{a}{b}$. We now obtain $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \cdot \tan y} = \dfrac{a}{1 - \frac{a}{b}} = \dfrac{ab}{b - a}$.

20. If $k = 0$ we have $\sin x + \cos x = 0 \rightarrow \tan x = -1 \rightarrow x = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$. This gives a sum of $\dfrac{5\pi}{2}$. If $k = 1$ we have $2\cos x = 0 \rightarrow x = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ for a sum of 2π. If $0 < k < 1$ we have $\sin x + \cos x = k\sin x - k\cos x \rightarrow (1 - k)\sin x = (-1 - k)\cos x \rightarrow \tan x = \dfrac{k + 1}{k - 1}$. Thus, $x = \mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right)$. Since $0 < k < 1$, $\dfrac{k + 1}{k - 1} < 0$. Therefore, the solutions between 0 and 2π are $\mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right) + \pi$ and $\mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right) + 2\pi$ for a sum of $2\mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right) + 3\pi$. As $k \rightarrow 0$, $\dfrac{k + 1}{k - 1} \rightarrow -1$ so $2\mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right)$ approaches $-\dfrac{\pi}{2}$ and the sum approaches $\dfrac{5\pi}{2}$. As $k \rightarrow 1$, $\dfrac{k + 1}{k - 1} \rightarrow -\infty$ and $2\mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right)$ approaches $-\pi$ and so the sum approaches 2π. Thus for $0 < k < 1$ the sum of the solutions lies between 2π and $\dfrac{5\pi}{2}$. For $k > 1$, $\dfrac{k + 1}{k - 1} > 0$ and the solutions between 0 and 2π are $\mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right)$ and $\mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right) + \pi$ for a sum of $2\mathrm{Tan}^{-1}\left(\dfrac{k + 1}{k - 1}\right) + \pi$. This is clearly less than or equal to $2 \cdot \dfrac{\pi}{2} + \pi = 2\pi$. So the largest possible sum of the solutions is $\dfrac{5\pi}{2}$. Squaring both sides will introduce extraneous solutions that can lead to answers such as 3π.

21. Let x = length of the external tangent. From the diagram $x^2 + (R - r)^2 = (R + r)^2 \rightarrow x^2 = 4Rr$. The external tangent will have integral length if Rr is a perfect square. There are 10 perfect squares from 1 to 100. But Rr can't equal 1 since $R \neq r$. If $Rr = 4$, then $x = 4$ and we have a 3-4-5 triangle formed. If $Rr = 9$, then $x = 6$ and we have a 6-8-10 triangle. The rest of the values for Rr work as well making for 10 – 1 answers.



22. Place a congruent figure on top, producing a cylinder with diameter 4 and height $6 + 9 = 15$. Its lateral surface area is $(2\pi \cdot 2) \cdot 15 = 60\pi$, so the lateral surface area of the given figure is half of that, namely 30π. The area of the base is 4π. The elliptical top has a major axis $AD = 5$ since there's a 3-4-5 triangle present and the minor axis is just the diameter of the base, 4. The ellipse's area is $\pi \cdot \frac{5}{2} \cdot 2 = 5\pi$. So the total surface area is 39π.



23. Replacing any two numbers with their sum leaves the sum of all *n* numbers invariant, so the sum of all the numbers is just the last number, namely 162. Thus, $\left(\frac{a_1 + \left[a_1 + (n-1) \cdot 1\right]}{2}\right) n = 162 \rightarrow \left(2a_1 + (n-1)\right) n = 324$. Both *n* and a_1 are positive integers so the left side represents a factorization of 324. Choosing values for *n* we find that if $n = 1$, $a_1 = 162$, if $n = 2$, then $2a_1 + 1 = 162$, but that fails since a_1 must be an integer. We obtain the following ordered pairs for (n, a_1): (3, 53), (4, 39), (6, 49/2), (9, 14), (12, 8), (18, 1/2), and for the rest of the values of *n*, a_1 is negative. The least value of a_1 is 8.

24. Let $m = 4p$. Then the vertex of $y = \frac{x^2}{4p} - k$ is (0, –*k*) and the focal point is $(0, -k + p)$. Then $\frac{m}{4} - k = 1 \rightarrow m = 4k + 4$. From $1 \le 4k + 4 \le 2010$ we obtain $-\frac{3}{4} \le k \le 501.5 \rightarrow 0 \le k \le 501$. Thus, there are 502 integer values of *k*.

25. Notice that $k^2 = a_1 + a_2 + a_3 + a_4 + a_5 = 5a_3$. Since $5a_3$ is a perfect square, $a_3 = 5b^2$ for some positive integer *b*. Hence, $k^2 = 5 \cdot 5b^2 \rightarrow k = 5b$. Since $k > a_5$ then $5b > a_5$. Now $a_5 = a_3 - 12$ so $5b > 5b^2 - 12$, giving $5b^2 - 5b - 12 < 0$. By inspection we see that the inequality is satisfied for $b = 1$ or 2. If $b = 1$, $a_3 = 5 \rightarrow a_2 = 11$. If $b = 2$, $a_3 = 20 \rightarrow a_2 = 26$. Answer: 26