46th ANNUAL

MASSACHUSETTS MATHEMATICS OLYMPIAD

2009 – 2010

A High School Competition Conducted by

THE MASSACHUSETTS ASSOCIATION
OF MATHEMATICS LEAGUES (MAML)

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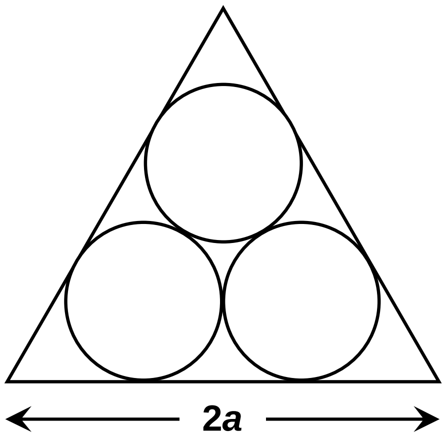
THE ACTUARIES' CLUB OF BOSTON

SECOND-LEVEL EXAMINATION

Tuesday, March 2, 2010

1. **SNUG CIRCLES (14 Points)**

a) Suppose that three circles of the same size are mutually tangent and snugly fit into an equilateral triangle of side $2a$, as shown. What is the area covered by the three circles? Show the work that produced your answer. (7 points)



b) Now find three noncongruent, nonoverlapping circles that cover more of the area inside the same triangle. Verify that the covered area is indeed greater than the covered area in part (a). (7 points)

2. **LOGS? ABSOLUTELY! (11 Points)**

Show that for every $a > 0$ and $b > 0$ (with the further proviso that neither a nor b can equal 1), $\left|\log_a(b)+\log_b(a)\right| \geq 2$.

3. I KNOW THE ANSWER, WHAT WAS THE QUESTION? (12 Points)

On a multiple-choice test one of the questions was illegible. The choice of answers was:

a) All of the below

b) None of the below

c) All of the above

d) Exactly one of the above

e) None of the above

f) None of the above

What was the correct answer? Justify your response.

4. A FUNCTIONAL INEQUALITY (12 Points)

Let f and g be any real-valued functions defined on [0,1]. Prove that there exist

a and b in [0,1] such that $\left|f(a)+g(b)-ab\right| \geq \frac{1}{4}$.

5. A RATIO OF AREAS (18 Points)

Given: triangle *AOB* with *A* (1, 2), *B* (4, 0), and *O* (0, 0). Think of *A* and *B* as $A_1(a_1, b_1)$ and $B_1(c_1, d_1)$, respectively, and define the coordinates of A_2 and B_2 as $(a_1 + b_1, a_1 - b_1)$ and $(c_1 + d_1, c_1 - d_1)$, respectively. Further, define the coordinates of A_{n+1} and B_{n+1} as $(a_n + b_n, a_n - b_n)$ and $(c_n + d_n, c_n - d_n)$, respectively.

a) Find the ratio of the area of triangle A_9OB_9 to the area of triangle A_4OB_4. (4 points)

b) For $n = 2k + 1$, find a formula for A_{2k+1}. (2 points)

c) For $n = 2k$, find, **with proof**, a formula for A_{2k}. (4 points)

d) For $n = 2k + 1$, find a formula for B_{2k+1}. (2 points)

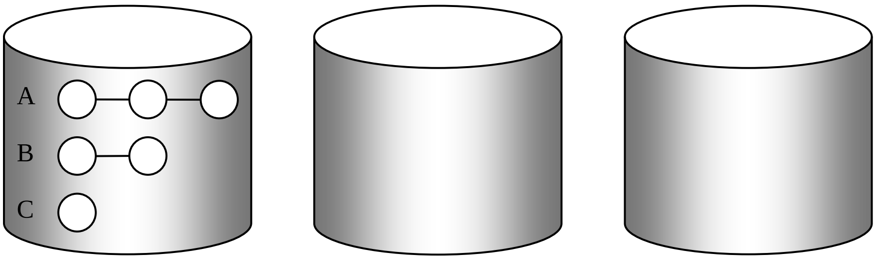
e) For $n = 2k$, find a formula for B_{2k}. (2 points)

f) Find, in terms of *n*, a formula for the area of the triangle with coordinates $O(0,0)$, A_n, and B_n. Prove your result using the formulas from (a) – (d). (4 points)

6. BIN THERE, DONE THAT (15 Points)

Sam has three bins, Bin 1, Bin 2, and Bin 3, the first of which contains three strings of balls, String 1, String 2, and String 3. He selects a string at random by reaching into Bin 1, grabbing a ball, and pulling out that ball and the entire string connected to it. Each ball has an equal probability of being selected, but each string does not. Sam then places this string of balls into Bin 2. In the same manner, he selects at random one of the two strings remaining in Bin 1 and places that one into Bin 2 as well. Finally, again in the same manner of selection, he selects at random one of the two strings of balls in Bin 2 and places it into Bin 3.

a) Let String A consist of 3 balls, String 2 consist of 2 balls, and String C consist of 1 ball, as depicted below. When following the selection process described above, what is the probability that String 1 is placed into Bin 3? (5 points)



Bin 1 **Bin 2** **Bin 3**

b) If String A consists of *a* balls, String B consists of *b* balls, and String C consists of *c* balls, what is the probability, in terms of *a*, *b*, and *c*, that String A is placed into Bin 3? (5 points)

c) Suppose that the numbers of balls on the strings form an arithmetic series, such that String A consists of a balls, String 2 consists of $a + d$ balls, and String C consists of $a + 2d$ balls. Find, in simplified form, the probability that String B is placed into Bin 3. (5 points)

7. **THE SAME SHADE (18 Points)**

In the figure, ΔCDE is a right triangle, $ABCD$ and $CEFG$ are squares, and I is the intersection of $\overline{AE}$ and $\overline{DF}$. Show that the shaded regions in the triangle have the same area.