# FORTY-FIFTH ANNUAL OLYMPIAD

# HIGH SCHOOL PRIZE COMPETITION

# IN MATHEMATICS

**2008 – 2009**

**Conducted by**

**The Massachusetts Association**
**of**
**Mathematics Leagues**
**(MAML)**

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**The Actuaries' Club of Boston**

**SECOND LEVEL EXAMINATION**

# SOLUTIONS

**Tuesday, March 3, 2009**

1. WARMUP

   (a) In $\Delta ABC$, let the bisector of $\angle A$ meet the circumcircle of the triangle at the point $Q$. Let $L$ be the line tangent to the circumcircle at $Q$. Prove that $L$ is parallel to $BC$.

   (b) Let two equal circles, $C_1$ and $C_2$, be externally tangent and tangent to line $L$ (on the same side of $L$). Let smaller circle, $C_3$, between $C_1$, $C_2$, and $L$, be tangent to all three. Let $O_1$ be the center of $C_1$, $O_3$ be the center of $C_3$, and let $P$ be the point of tangency of $C_1$ and $C_2$. Prove that $O_1P\,O_3$ is a 3-4-5 right triangle.

Solution (a):

Proof: Let $O$ be the center of the circumcircle of $\Delta ABC$. Then $OQ$ is a radius of the circumcircle and is therefore perpendicular to $L$. Also, since $AQ$ is the bisector of angle $A$, $\overset{\frown}{CQ} = \overset{\frown}{BQ}$. Therefore the radius $OQ$ is on the perpendicular bisector of chord $BC$. Since $OQ$ is perpendicular to both $BC$ and $L$, $L$ is parallel to $BC$.

Solution (b):

Proof: Let $C_1$ and $C_2$ have radius $R$ and let $C_3$ have radius $X$. Then the legs of right triangle $O_1P\,O_3$ have length $R$ and $R - X$, and the hypotenuse has length $R + X$. Therefore,

$$R^2 + (R - X)^2 = (R + X)^2$$

Solving, we have $X = \frac{R}{4}$. So the legs are $\frac{3R}{4}$ and $R$, and the hypotenuse is $\frac{5R}{4}$. The result follows.

2. FACTORIAL ZEROES

(a) In how many zeros does 2000! end?

(b) Find the smallest $n$ for which $n!$ ends in exactlty 2007 zeros.

(c) Find the smallest $N$ (or show that none exists) for which the decimal representation of $N!$ ends in exactly 2007 zeros.

SOLUTION:

(a) **To be inserted here**

(b) **To be inserted here**

(c) The answer is 8040. Because there is one zero at the end of $N!$ for each factor of $10 = 2\times 5$, if $N! = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \ldots$, the number of zeros is the smaller of $a$ and $c$. Since a (far) exceeds $c$ (why?), the number of zeros at the end of $N!$ equals the number of 5's in the prime factorization of $N!$ This number, $Z(N)$, is $\frac{N}{5}+\frac{N}{5^2}+\frac{N}{5^3}+\ldots$. This is because each number less than or equal to $N$ of the form $M5^k$, where $M$ is not divisible by 5, is counted exactly $k$ times in this sum. So if there is an $N$ for which $Z(N)=2007$, then

$$2007 < \frac{N}{5}+\frac{N}{5^2}+\frac{N}{5^3}+\ldots = \frac{N}{4}.$$

Thus $N > 8025$. Noting that $Z(8025)=Z(8026)=Z(8027)=Z(8028)=Z(8029)$, using the formula we find that

$Z(8030)=Z(8031)=Z(8032)=Z(8033)=Z(8034)=$

$1606+321+64+12+2=2005.$

Thus $Z(8035)=2006$, and finally $Z(8040)=2007$.

(Challenge: Note that no factorial ends in exactly 5 zeros. How many positive integers less than 2007 are *not* the exact number of zeros at the end of any factorial?)

3. (a) WHAT ARE THE ODDS?

There are twenty containers. One of them contains ten balls numbered 1 through 10. The other nineteen each contain 10,000 balls numbered 1 through 10,000. A container is selected at random and a ball is randomly selected from it. The ball is number 8. What is the probability that the container picked was the one with ten balls?

(b) QUADRATIC COEFFICIENTS

Suppose that the quadratic $Ax^2 + Bx + C$ has no real zeros and that $A + B + C > 0$. Prove that C > 0.

SOLUTION:

(a) Before drawing the ball there is a $\frac{1}{20}$ probability that the container with ten balls was picked, and the probability that ball number 8 is then drawn is $\frac{1}{10}$. Before drawing the ball there is a $\frac{19}{20}$ probability that a container containing 10,000 was picked, and the probability that ball number 8 is then drawn is $\frac{1}{10,000}$.

So the probability that ball number 8 was drawn is

$$\frac{1}{20}\cdot\frac{1}{10}+\frac{19}{20}\cdot\frac{1}{10,000}=\frac{1,019}{200,000}$$

and the probability that the container with ten balls was picked, given that ball number 8 was drawn is

$$\frac{\frac{1}{20}\cdot\frac{1}{10}}{\frac{1}{20}\cdot\frac{1}{10}+\frac{19}{20}\cdot\frac{1}{10,000}}=\frac{1,000}{1,019}\cong 0.981 \text{ or just over 98\%.}$$

(b) Let $P(x) = Ax^2 + Bx + C$, and note that $P(1) = A + B + C > 0$. Because P has no real zeros, it follows that $P(x) > 0$ for all real *x*. But $P(0) = C$, and therefore $C > 0$.

4. PRIME PROBLEMS

(a) A pair of prime numbers is a *twin prime* pair if they differ by 2. For example, 3 and 5, 5 and 7, and 41 and 43 are twin prime pairs. Prove that if $P_1$ and $P_2$, each greater than 3, is a twin prime pair, then $P_1 + P_2$ is divisible by 12.

(b) Two prime numbers are *successive* if there are no primes between them. For example 11 and 13, 19 and 23, and 47 and 53 are pairs of successive primes. Prove that the sum of two successive primes cannot be the product of exactly two primes.

(c) Find, with reasons, the largest integer that ends in 4 that is not the sum of two odd composite integers.

(d) Find, with reasons, the largest even integer that cannot be written as the sum of two odd composite integers.

SOLUTION:

(a) Any odd number is of the form $6k-1$ or $6k+1$ or $6k+3$. Since $6k+3$ is a multiple of 3, a twin prime pair, each greater than 3, must be of the form $6k-1$ and $6k+1$. So their sum is $12k$.

(b) The primes 2 and 3 are consecutive, and their sum is not the product of exactly two primes. The sum of two consecutive odd primes is twice their average $A$, and $A$ is a whole number which, by the definition of consecutive primes, is not prime. Therefore $2A$ cannot be the product of exactly two primes.

(c) **Insert Solution Here**

(d) **Insert Solution Here**

5. PHANTASMAGORIC FRACTALS IN FOUR DIMENSIONS:

To create a fractal in two dimensions:

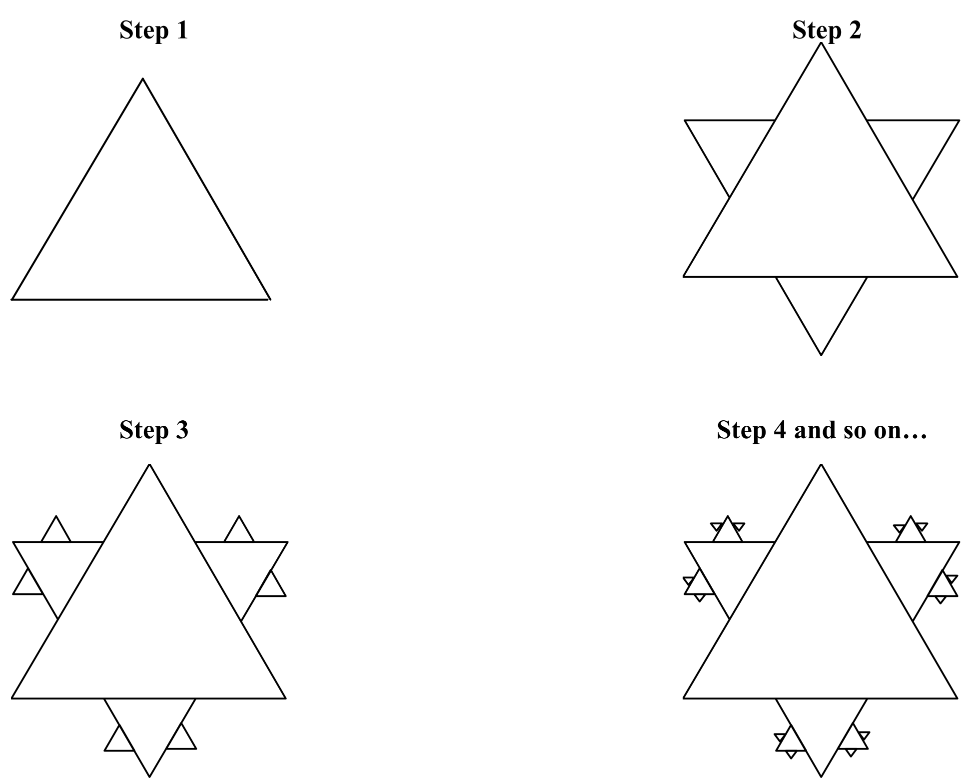
(1) Start with an equilateral triangle with area 1.

(2) On each exterior side of the equilateral triangle, construct a new equilateral triangle such that one side of each new triangle lies on one side of the original triangle and connects the trisection points.

(3) Now, on each exposed side of the triangles constructed, create a new equilateral triangle such that one side of each new triangle is a segment of one side of a previous triangle and has one third the side length of the previous triangle.

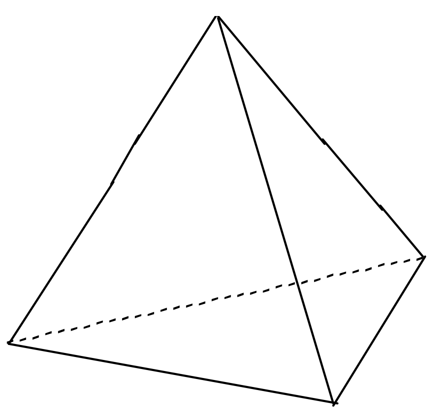
(4) Repeat step 3 indefinitely.

Note that the first covering uses all 3 sides of the original triangle while the rest of the coverings use only 2 sides of the added triangles. The process is outlined in the figure below.

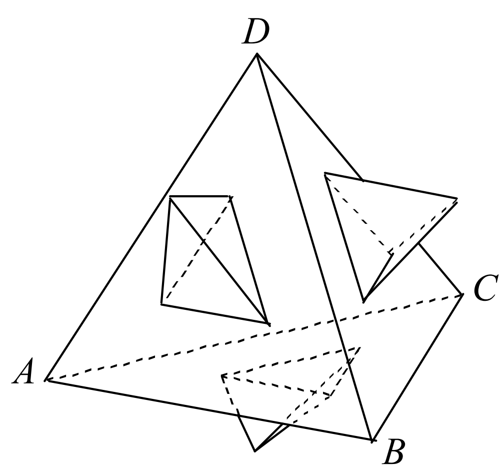


(a) If the process is continued infinitely, determine the area of the figure.

(b) Imagine that you have a tetrahedron of volume 1 and are adding tetrahedrons in the same manner as you added triangles in the steps above, that is, you place tetrahedrons on the center of each face of the original tetrahedron so that each side of the new tetrahedron is one-third the length of each side of the original tetrahedron as illustrated in the figure below. Note that the first covering uses all 4 faces of the original tetrahedron while all the additional coverings only use 3 faces of the created tetrahedrons. Also, the tetrahedron on face *ADC* in the figure below is not shown. What is the volume of the three-dimensional object you created as you add tetrahedrons using this method forever?



**Step 1**




**Step 2 and beyond**

c) Now imagine that you are in four dimensional space staring at a hyper-tetrahedron, also known as a pentatope. It has 5 "faces" each of which is a tetrahedron. Suppose that you are attaching pentatopes to the faces of the pentatope in the same manner in which you attached tetrahedrons to the faces of the tetrahedrons in (b). The original pentatope has a hyper-volume of 1 and every time you attach a pentatope to a "face" its edge length is 1/3 the edge length of the larger pentatope to which it is being attached. Pentatopes will be attached to all 5 "faces" of the original pentatope, but subsequent attachments will be to only 4 of the "faces" of the created pentatopes. What is the hyper-volume of the figure you create as you add pentatopes forever?

SOLUTION:

(a) The figure below represents the fractal created by the series of steps described in the problem. The figure is only partially complete in drawing but for our purpose we will imagine the fractal follows the process described in the steps thoroughly, that is, the triangles are added infinitely.

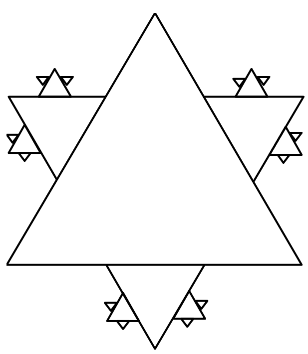


**Figure 1**

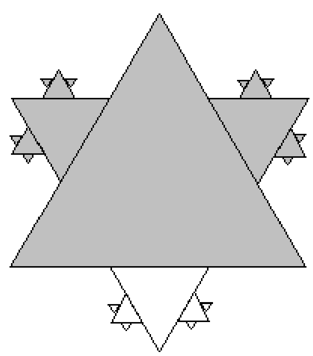


**Figure 2**

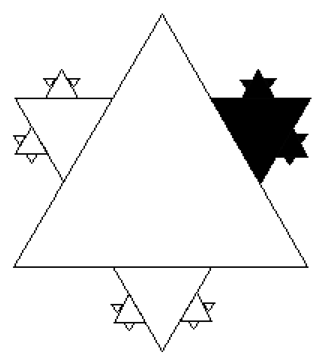


**Figure 3**

Let the gray shaded area in Figure 2 be represented by $A$. The dark area in Figure 3 would therefore be $\frac{A}{9}$, since the shape was created with an initial side length equal to $\frac{1}{3}$ the side length of the previous shape and a similarity in the ratio of $1:3^2$ thus exists between the dark area in Figure 3 and the gray area in Figure 2. Since the area of the original equilateral triangle is 1, the total shaded area $A$ can be represented by the following equation: $A = \frac{A}{9} + 1 + \frac{A}{9}$. This can be solved easily for $A$: $A = \frac{9}{7}$. Now, we can consider the unshaded area in Figure 2. The dark region in Figure 3 has the same area as the white region in Figure 2, so the white area equals $\frac{A}{9} = \frac{1}{7}$. Now, to find the total area, we add the grey shaded area to the white, which gives $\frac{9}{7} + \frac{1}{7} = \frac{10}{7}$.

This problem can also be solved using geometric series. The large triangle has an area of 1, each of the second-largest triangles has an area of $\frac{1}{3^2}$, each of the third-largest triangles has an area of $\left(\frac{1}{3^2}\right)^2 = \frac{1}{3^4}$, etc. There is one large triangle, 3 second-largest triangles, 6 third-largest triangles, and so on. The sum of the areas is

$$1 + 3 \cdot \frac{1}{3^2} + 6 \cdot \frac{1}{3^4} + 12 \cdot \frac{1}{3^6} + \ldots = 1 + \sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{9}\right)^n = 1 + \frac{\frac{1}{3}}{1 - \frac{2}{9}} = 1 + \frac{3}{7} = \frac{10}{7}.$$

(b) To find the volume of the tetrahedron fractal, the same method above is used. We ignore one figure or spiny projection coming out of the complete figure, for example, that projection that comes out of the base, and concentrate on the rest of the volume, $V$, consisting of 3 fractal tetrahedrons. The ratio between one spiny projection and $V$ is $1:3^3$ (since the ratio of the sides of the tetrahedron to the earlier tetrahedron is 1:3). We can use this to set up an equation to solve for $V$: $V = \frac{V}{27} + 1 + \frac{V}{27} + \frac{V}{27}$; this simplifies to $V = \frac{9}{8}$. We can then find the volume of the spiny projection we ignored by evaluating $\frac{V}{27}$, which equals $\frac{9}{8} \div \frac{1}{27} = \frac{1}{24}$. Now, to find the total volume, we evaluate $V + \frac{1}{24}$, which equals $\frac{9}{8} + \frac{1}{24} = \frac{7}{6}$.

This problem can also be solved using geometric series. The large tetrahedron has a volume of 1, each of the second-largest tetrahedrons has a volume of $\frac{1}{3^3}$, each of the third-largest tetrahedrons has a volume of $\left(\frac{1}{3^3}\right)^2 = \frac{1}{3^6}$, etc. There is one large tetrahedron, 4 second-largest tetrahedrons, 12 third-largest tetrahedrons, and so on. The sum of the volumes is

$$1 + 4 \cdot \frac{1}{3^3} + 12 \cdot \frac{1}{3^6} + 36 \cdot \frac{1}{3^9} + \ldots =$$

$$1 + \sum_{n=0}^{\infty} \frac{4}{27} \cdot \left(\frac{3}{27}\right)^n = 1 + \frac{\frac{4}{27}}{1 - \frac{3}{27}} = 1 + \frac{4}{24} = \frac{7}{6}.$$

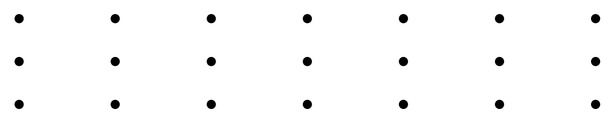
(c) To find the hyper-volume, we follow the same method above. Disregard one extension of the hyper-tetrahedron and let the rest of the hyper-volume be represented by *H*. Then, the ratio of one extension to *H* is $1:3^4$ because the ratio of the sides is 1:3. Then, we solve for *H* by using $H = \frac{H}{81} + 1 + \frac{H}{81} + \frac{H}{81} + \frac{H}{81}$, so $H = \frac{81}{77}$. We then find the hyper-volume of the extension we ignored by finding $\frac{H}{81}$, which is $\frac{1}{77}$. We then find the hyper-volume by computing $H + \frac{1}{77}$, which equals $\frac{82}{77}$.

This problem can also be solved using geometric series. The large pentatope has a hyper-volume of 1, each of the second-largest pentatopes has a hyper-volume of $\frac{1}{3^4}$, each of the third-largest pentatopes has a hyper-volume of $\left(\frac{1}{3^4}\right)^2 = \frac{1}{3^8}$, etc. There is one large pentatope, 5 second-largest pentatopes, 20 third-largest pentatopes, and so on. The sum of the hyper-volumes is

$$1 + 5 \cdot \frac{1}{3^4} + 20 \cdot \frac{1}{3^8} + 80 \cdot \frac{1}{3^{12}} + \ldots = 1 + \sum_{n=0}^{\infty} \frac{5}{81} \cdot \left(\frac{4}{81}\right)^n = 1 + \frac{\frac{5}{81}}{1 - \frac{4}{81}} = 1 + \frac{5}{77} = \frac{82}{77}$$

## 6. COLOR THE DOTS

Each of the 21 dots in the array below is to be colored with one of two colors. Prove that, no matter how the coloring is done, there will be four dots of the same color that form the vertices of a rectangle.



SOLUTION:

First note that of the three dots in each column (at least) two of them must have the same color. Also note that there are $2^3 = 8$ different ways that a column can be colored. There are then two cases to consider: either two columns are identically colored, or no two columns are identically colored.

(I) If two columns are identically colored, then two dots of the same color and position in each column form the vertices of a rectangle.

(II) If no two columns are identically colored, then exactly 1 of the 8 possible colorings is not used. Therefore in (at least) one column all three dots are the same color -- say red. Because only one coloring has not been used, there will be (at least) one other column that contains two red dots (in fact, there will be at least two!). Those two red dots and the two red dots in the corresponding positions of the red column form the vertices of a rectangle.

7. A DOUBLE INEQUALITY

Prove that, for all positive integers n,

$$2\left(\sqrt{n+1}-1\right) < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}\,.$$

SOLUTION:

We use induction. The result is true for $n=1$ because

$$2\left(\sqrt{2}-1\right) < 1 < \sqrt{2}\,.$$

Now assume that

$$2\left(\sqrt{k+1}-1\right) < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} < 2\sqrt{k}$$

for a positive integer k. We will show that

$$2\left(\sqrt{k+2}-1\right) < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

Our assumption implies that

$$2\left(\sqrt{k+1}-1\right) + \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$$

so it suffices to show that

(*) $\quad 2\left(\sqrt{k+2}-1\right) < 2\left(\sqrt{k+1}-1\right) + \frac{1}{\sqrt{k+1}}$ and

(**) $\quad 2\sqrt{k} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}\,.$

Inequality (*) is equivalent to $2\sqrt{(k+1)(k+2)} < 2\sqrt{k+1}+1$ and squaring both sides shows that this inequality holds. Inequality (**) is equivalent to

$2\sqrt{k(k+1)} < 2k+1$ and squaring both sides shows that this inequality holds.
So, by the Principle of Mathematical Induction, the result holds.

8. A MENAGERIE

Find the sum of all $\theta$, $0 \le \theta < 2\pi$, such that the graphs of

$f(x) = 2 \cdot \sin^2\theta \cdot x^2 + \cot\theta \cdot x - 1$ and $g(x) = 2 \cdot \cos^2\theta \cdot x^2 - \tan\theta \cdot x + \cot^2(2\theta)$

intersect at exactly one point.

SOLUTION:

Consider the difference function $h(x) = g(x) - f(x)$. The magnitude of $h$ represents the vertical distance between $f$ and $g$ at $x$. If $h$ equals 0, then $f$ and $g$ intersect. If $h(x)$ equals 0 for only one value of $x$, than $f(x)$ and $g(x)$ intersect exactly once. Thus, we have $h(x) = g(x) - f(x) = (2\cos^2\theta - 2\sin^2\theta)x^2 - (\tan\theta + \cot\theta)x + (\cot^2 2\theta + 1)$ with $\sin\theta \ne 0$ and $\cos\theta \ne 0$, meaning that $\theta \ne 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, or $2\pi$.

Since $\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} = \frac{1}{\frac{1}{2}\sin(2\theta)} = 2\csc(2\theta)$,

then $h(x) = (2\cos 2\theta)x^2 - (2\csc 2\theta)x + \csc^2 2\theta$.

If $\cos 2\theta \ne 0$, then $h(x)$ is quadratic. So $h(x)$ is quadratic as long as $2\theta \ne \frac{\pi}{2} + \pi k$, i.e., as long as $\theta \ne \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$. For $h(x)$ to equal 0 exactly once, the discriminant of the quadratic must equal 0. That is, $B^2 - 4AC = 0$, where $A = 2\cos 2\theta$, $B = 2\csc 2\theta$, and $C = \csc^2 2\theta$. To solve for $\theta$, $B^2 = 4AC \Rightarrow (2\csc 2\theta)^2 = 4(2\cos 2\theta)(\csc^2 2\theta)$.
Thus, $4\csc^2(2\theta) = 8(\csc^2 2\theta)\cos 2\theta$. Because $\csc(2\theta) \ne 0$, we may divide by a factor of $8 \cdot \csc^2(2\theta)$, leaving $\frac{1}{2} = \cos 2\theta$. Setting $2\theta = \frac{\pi}{3} + 2\pi k$ and $2\theta = \frac{5\pi}{3} + 2\pi k$ gives $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ in the interval [0, 2π].

If $\cos(2\theta) = 0$, then $h$ is linear and $h(x)$ equals 0 for exactly one $x$ if $B = 2\csc 2\theta \ne 0$. This gives p0 $\rightarrow$ $x = \frac{1}{2}\csc 2\theta$. Because $\csc 2\theta \ne 0$, then $h(x)$ equals 0 for exactly one value of $x$ if $\cos 2\theta = 0$, i.e. if $2\theta = \frac{\pi}{2} + \pi k$, giving $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$. Note that while we rejected these solutions when *h* was quadratic, we accept them when *h* is linear.

Hence, $h(x) = 0$ exactly once and, therefore, $f(x)$ and $g(x)$ intersect exactly once when $\cos 2\theta = \frac{1}{2}$ or $\cos 2\theta = 0$, giving

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$. These sum to $8\pi$.