FORTY-FIFTH ANNUAL OLYMPIAD

HIGH SCHOOL PRIZE COMPETITION

IN MATHEMATICS

2008 – 2009

Conducted by

The Massachusetts Association
of
Mathematics Leagues
(MAML)

Sponsored by

The Actuaries' Club of Boston

SECOND LEVEL EXAMINATION

Tuesday, March 3, 2009

1. WARMUP

(a) In ΔABC, let the bisector of $\angle A$ meet the circumcircle of ΔABC at point Q. Let $\overline{QT}$ be the line tangent to the circumcircle at Q. Prove that $\overline{QT}$ is parallel to $\overline{BC}$.

(b) Congruent circles C_1 and C_2 are externally tangent at P. Line ℓ is a common external tangent of C_1 and C_2. Circle C_3 is smaller than the other two circles and is tangent to C_1, C_2, and ℓ. Prove that ΔC_1C_3P is similar to a 3-4-5 triangle.

2. FACTORIAL ZEROES

(a) In how many zeros does 2000! end?

(b) Find the smallest n for which $n!$ ends in exactly 2007 zeros.

3. (a) WHAT ARE THE ODDS?

There are twenty containers. One of them contains ten balls numbered 1 through 10. The other nineteen each contain 10,000 balls numbered 1 through 10,000. A container is selected at random and a ball is randomly selected from it. The ball is number 8. What is the probability that the container picked was the one with ten balls?

(b) QUADRATIC COEFFICIENTS

Suppose that the quadratic $Ax^2 + Bx + C$ has no real zeros and that $A + B + C > 0$. Prove that $C > 0$.

4. PRIME PROBLEMS

(a) A pair of prime numbers is a *twin prime* pair if they differ by 2. For example, 3 and 5, 5 and 7, and 41 and 43 are twin prime pairs. Prove that if P_1 and P_2, each greater than 3, is a twin prime pair, then $P_1 + P_2$ is divisible by 12.

(b) Two prime numbers are *successive* if there are no primes between them. For example 11 and 13, 19 and 23, and 47 and 53 are pairs of successive primes. Prove that the sum of two successive primes cannot be the product of exactly two primes.

(c) Find, with reasons, the largest integer that ends in 4 that is not the sum of two distinct odd composite integers.

(d) Find, with reasons, the largest even integer that cannot be written as the sum of two distinct odd composite integers.

5. PHANTASMAGORIC FRACTALS IN FOUR DIMENSIONS:

To create a fractal in two dimensions:

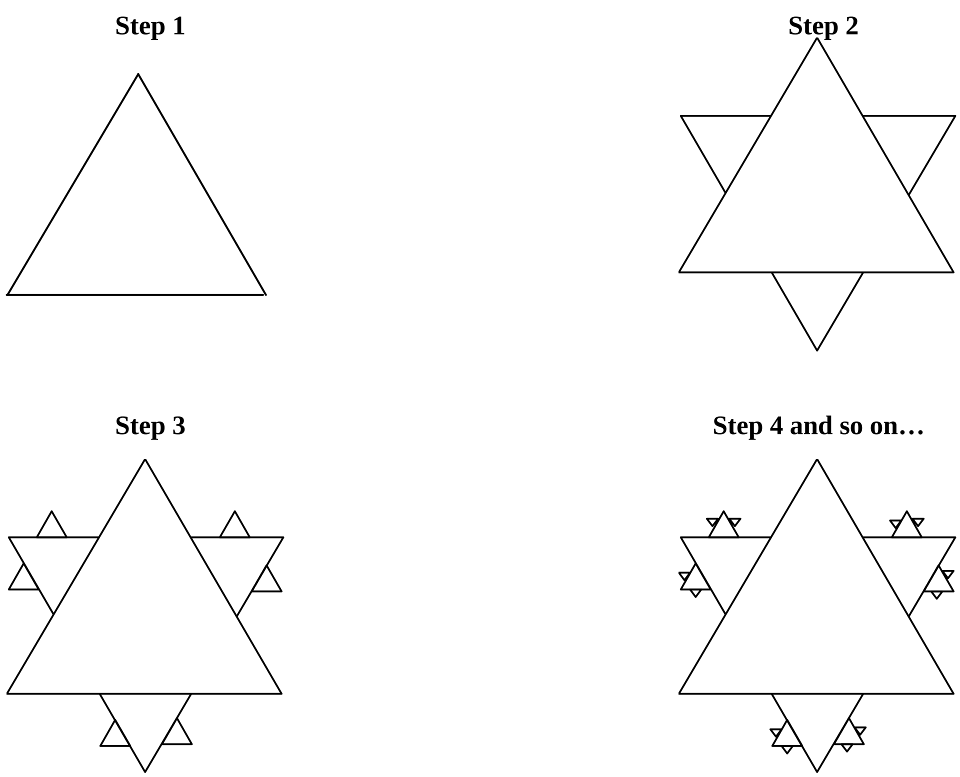
(1) Start with an equilateral triangle with area 1.

(2) On each exterior side of the equilateral triangle, construct a new equilateral triangle such that one side of each new triangle lies on one side of the original triangle and connects the trisection points.

(3) Now, on each exterior exposed side of the triangles constructed, create a new equilateral triangle such that one side of each new triangle is a segment of one side of a previous triangle and has one third the side length of the previous triangle.

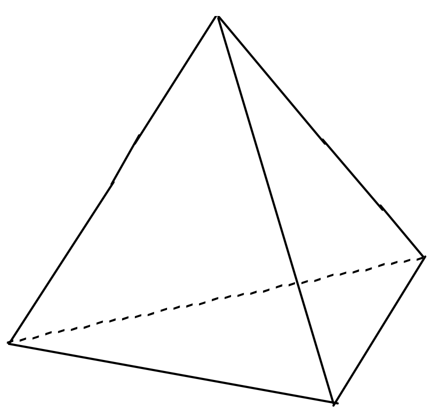
(4) Repeat step 3 indefinitely.

Note that the first covering uses all 3 sides of the original triangle while the rest of the coverings use only 2 sides of the added triangles. The process is outlined in the figure below.

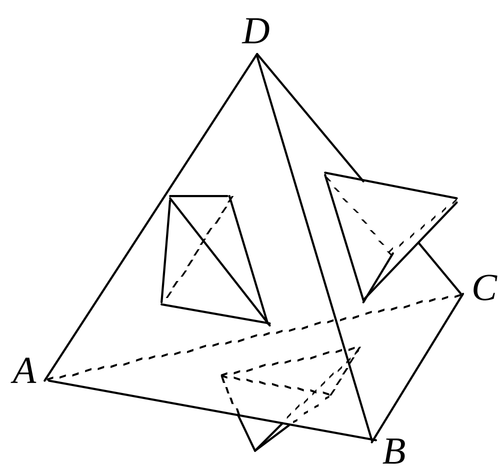


(a) If the process is continued infinitely, determine the area of the figure.

(b) Imagine that you have a tetrahedron of volume 1 and are adding tetrahedrons in the same manner as you added triangles in the steps above, that is, you place tetrahedrons on the center of each face of the original tetrahedron so that each side of the new tetrahedron is one-third the length of each side of the original tetrahedron as illustrated in the figure below. Note that the first covering uses all 4 faces of the original tetrahedron while all the additional coverings only use 3 faces of the created tetrahedrons. Also, the tetrahedron on face *ADC* in the figure below is not shown. What is the volume of the three-dimensional object you created as you add tetrahedrons using this method forever?



Step 1

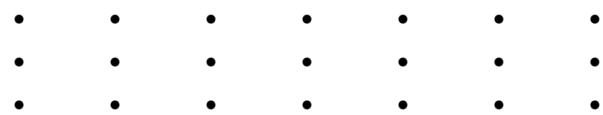


Step 2 and beyond

c) Now imagine that you are in four dimensional space staring at a hyper-tetrahedron, also known as a pentatope. It has 5 "faces" each of which is a tetrahedron. Suppose that you are attaching pentatopes to the faces of the pentatope in the same manner in which you attached tetrahedrons to the faces of the tetrahedrons in (b). The original pentatope has a hyper-volume of 1 and every time you attach a pentatope to a "face" its edge length is 1/3 the edge length of the larger pentatope to which it is being attached. Pentatopes will be attached to all 5 "faces" of the original pentatope, but subsequent attachments will be to only 4 of the "faces" of the created pentatopes. What is the hyper-volume of the figure you create as you add pentatopes forever?

6. COLOR THE DOTS

Each of the 21 dots in the array below is to be colored with one of two colors. Prove that, no matter how the coloring is done, there will be four dots of the same color that form the vertices of a rectangle.



7. A DOUBLE INEQUALITY

Prove that, for all positive integers n,

$$2\left(\sqrt{n+1}-1\right) < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}\,.$$

8. A MENAGERIE

Find the sum of all θ, $0 \le \theta < 2\pi$, such that the graphs of

$f(x) = 2 \cdot \sin^2\theta \cdot x^2 + \cot\theta \cdot x - 1$ and $g(x) = 2 \cdot \cos^2\theta \cdot x^2 - \tan\theta \cdot x + \cot^2(2\theta)$

intersect at exactly one point.