Forty-Fourth Annual

MASSACHUSETTS MATHEMATICS OLYMPIAD

2007–2008

Conducted by

The Massachusetts Association of Mathematics Leagues

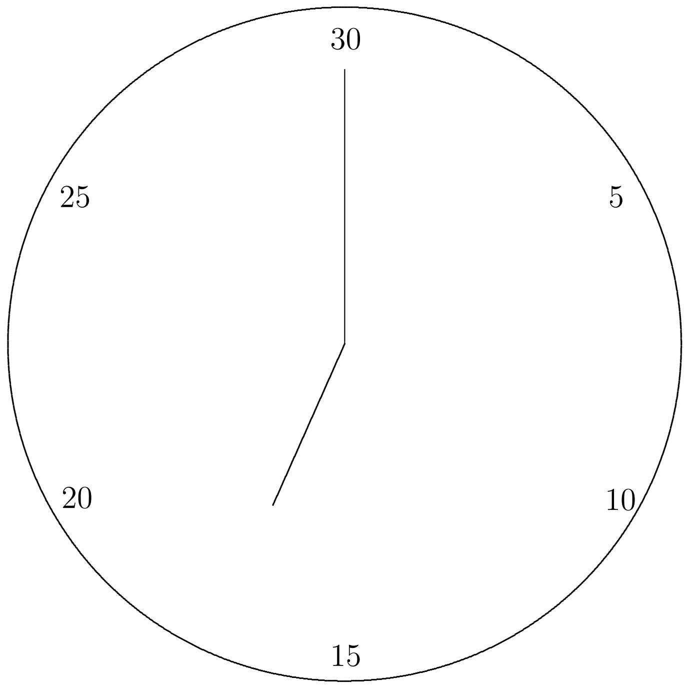
Sponsored by

The Actuaries' Club of Boston

SECOND LEVEL SOLUTIONS

Tuesday, March 4, 2008

1. **Time to begin.** As you know, a day is divided into 24 hours, each with 60 minutes. A confused watchmaker believes that a day has 60 hours, each with 24 minutes, and builds a fully-functional and accurate watch as follows:



The watch shows 17 o'clock. As with a standard watch, the numbers on the outside represent the hour; the hour hand makes two revolutions per day; and at "noon" and "midnight" (both 30:00 on the confused watch), both the hour and the minute hands point directly upward.

(a) (4 points) What will be the acute angle, in degrees, formed by the hands of the confused watch in 17 minutes — when it reads 17:17?

(b) (3 points) What time on the confused watch corresponds to 8:00 PM on a regular watch?

(c) (4 points) Between noon and midnight, how many times will the hands on the confused watch be perpendicular?

(d) (4 points) Suppose this watch were placed beside a regular watch (also fully functional and accurate). Between noon and midnight, how many times would the hands of the two watches point in the same directions? Exclude both noon and midnight.

Solution:

For this entire solution, we differentiate between a regular watch and the confused watch by referring to "regular hours" and "regular minutes" versus "confused hours" and "confused minutes."

(a) As with a standard watch, the confused hour hand travels 360° in half a day, so it travels $\frac{360°}{30} = 12°$ per confused hour.

At 17:17, the confused minute hand will be at $\frac{17}{24}(360°) = 255°$.

At 17:00, the confused hour hand was at $\frac{17}{30}(360°) = 204°$. Over the next 17 minutes it traveled an additional $\frac{17}{24}(12°) = 8.5°$, placing it at 212.5°.

Thus the angle formed is $255 - 212.5 = \boxed{42.5°.}$

(b) 8:00 PM is $8 \cdot 60 = 480$ minutes after noon. Because there are 24 minutes per confused hour, this represents 20 confused hours. So the confused watch reads $\boxed{\text{20:00 PM.}}$

(c) Consider the same question for a traditional watch. Because the minute hand travels much faster than the hour hand, we expect the hands to be perpendicular twice per hour — more specifically, once in each half-hour period, between x:00 and $x : 30$ as well as between x:30 and $(x+1)$:00. But at 3:00 PM and at 9:00 PM, the perpendicular hands straddle these half-hour periods, removing two times. Hence for a standard watch the answer is $24 - 2 = 22$ times.

For a confused watch, again we anticipate that the hands will be perpendicular once every confused half-hour, between x:00 and x:12 as well as between x:12 and $(x+1)$:00, of which there are 60 (recall that there are 24 confused minutes in a confused hour). But again, there are two exceptions: one at 7:12 (which is half-way between 7:00 and 8:00) and one at 22:12 (which is half-way between 22:00 and 23:00). In both cases, the confused minute hand points straight down and the confused hour hand points either directly left or directly right. Hence the answer is $60 - 2 = \boxed{\text{58 times.}}$

Alternately, the minute hand makes 30 revolutions while the hour hand makes 1 revolution, so there are 29 relative revolutions. For each relative revolution, there are two times when the hands are perpendicular. Hence the answer is $\boxed{\text{58 times.}}$

(d) As discussed in (a), the regular hour hand and the confused hour hand both travel 360° in half a day. Since the hour hands always agree, it is only necessary for the minute hands to agree.

In half a day, the confused minute hand makes 30 revolutions while the regular minute hand makes 12 revolutions. So the confused minute hand ends the half-day period 18 "laps" ahead, because it passes the other hand 18 times — the last time, at midnight. Hence excluding both noon and midnight, the minute hands agree $\boxed{\text{17 times.}}$

2. **Studying hard.** A study hall is held in a classroom with a single row of n desks. To keep the students quiet, the strict teacher creates two rules: (1) no two students can sit in adjacent desks, and (2) once sitting no student can move.

   Students begin sitting, and after a certain amount of time, the study hall is "full" — in other words, no more students can sit without violating rules 1 or 2.

   For the following questions, when giving any diagrams, use "X" for a filled seat and "O" for an empty seat.

   (a) (2 points) Suppose $n = 15$. What is the smallest number of students who could be in the study hall when it is "full"? What is the largest number of students who could be in the study hall when it is "full"? Show an example of both cases.

   (b) (3 points) For each of $n = 1$ to $n = 7$, how many different ways can the study hall be "filled"? Each seat in the row is distinct (so two seating arrangements that are mirror images should both be counted).

   (c) (1 points) The number of seating arrangements for a certain number of desks n is a function of the number of seating arrangements for two smaller values of n. Find this recursive relationship. (There are two correct formulas possible. You need only state one.)

   (d) (2 points) Find the number of seating arrangements when $n = 15$.

   (e) (7 points) Prove the recursive relationship you identified in (c).

Solution:

(a) The smallest number of students is $\boxed{5:}$

$$OXOOXOOXOOXOOXO$$

The largest number of students is $\boxed{8:}$

$$XOXOXOXOXOXOXOX$$

(b)

seats	arrangements
1	1
2	2
3	2
4	3
5	4
6	5
7	7

(For reference, the ways are:

$$X$$
$$XO, OX$$
$$XOX, OXO$$
$$XOOX, XOXO, OXOX$$
$$XOOXO, OXOXO, XOXOX, OXOOX$$
$$XOXOXO, OXOOXO, XOOXOX, XOXOOX, OXOXOX$$

and

$XOOXOXO, XOXOOXO, OXOXOXO, XOOXOOX, OXOXOOX, XOXOXOX, OXOOXOX$

(c) $\boxed{S(n) = S(n-2) + S(n-3),\ n \geq 4 \textbf{ OR } S(n) = S(n-1) + S(n-5),\ n \geq 6.}$ Either works. Only one need be provided.

(d) Following the progression... 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, $\boxed{65.}$

(e) **Proof that** $S(n) = S(n-2) + S(n-3)$**:**

(i) $S(n) \geq S(n-2) + S(n-3)$. Every sequence ends in X or O. If it ends in X, it ends in OX, because XX is impossible (two students cannot sit together). If it ends in O, it ends in XO, because OO is also impossible (another student could sit).
To each sequence of $n-2$ seats, we can always add OX, for it results in $\ldots OX[OX]$ or $\ldots XO[OX]$, both of which introduce no problems.
To each sequence of $n-3$ seats, we can always add OXO, for it results in $\ldots OX[OXO]$ or $\ldots XO[OXO]$, both of which also introduce no problems.
Because the $(n-2)$-seat arrangements become n-seat arrangements ending in X, and the $(n-3)$-seat arrangements become n-seat arrangements ending in O, the two groups do not overlap.
So $S(n) \geq S(n-2) + S(n-3)$.

(ii) $S(n) \leq S(n-2)+S(n-3)$. If a sequence ends in X, it ends in OX. In this case it either ends in $XOOX$ or XOX. In either case, we can remove the last two seats without introducing problems.
If a sequence ends in O, it ends in XO. In this case it either ends in $XOOXO$ or $XOXO$. In either case we can remove the last three seats without introducing problems.
Since removing three seats and removing two seats results in different numbers of seats, the resulting seat arrangements do not overlap.
So $S(n) \leq S(n-2)+S(n-3)$.

(iii) Since $S(n) \geq S(n-2)+S(n-3)$ and $S(n) \leq S(n-2)+S(n-3)$, we have $S(n) = S(n-2)+S(n-3)$.

Proof that $S(n) = S(n-1)+S(n-5)$**:**

(I find this one much harder to see.)

(i) $S(n) \geq S(n-1)+S(n-5)$. Every sequence ends in X or O. If it ends in X, it ends in OX, because XX is impossible (two students cannot sit together). If it ends in O, it ends in XO, because OO is also impossible (another student could sit).
To each sequence of $n-5$ seats, we can always add $OXOXO$, for it results in $\ldots XO[OXOXO]$ or $\ldots OX[OXOXO]$, both of which introduce no problems. So we can always create an n seat sequence from an $n-5$ seat sequence.
To each sequence of $n-1$ seats, if the sequence ends in OXO, we can add X to get $OXO[X]$; if it ends in OOX, we can add O to get $OOX[O]$; and if it ends in XOX we can change the ending to $XO[O]X$. In all three cases these additions cause no problems (two students seated beside each other, or an open seat). So we can always create an n seat sequence from an $n-1$ seat sequence.
Since the $(n-5)$-seat arrangements become n seat arrangements ending in $OXOXO$, and the $(n-1)$-seat arrangements all end with X or $OOXO$, the resulting seat arrangements do not overlap.
So $S(n) \geq S(n-1)+S(n-5)$.

(ii) $S(n) \leq S(n-1)+S(n-5)$. If a sequence ends in $OXOXO$, it either ends in $XOXOXO$ or $XOOXOXO$. In either case, removing the last five seats causes no problems.
If a sequence ends XOX, we can remove the final X without problems. If it ends with $XOXO$, we can remove the final O without problems. If it ends with $XOOX$, we can remove one of the O's without problems. In each case, removing one seat causes no problems.
Since removing five seats and removing one seat results in different numbers of seats, the resulting seat arrangements do not overlap.
So $S(n) \leq S(n-1)+S(n-5)$.

(iii) Since $S(n) \geq S(n-1)+S(n-5)$ and $S(n) \leq S(n-1)+S(n-5)$, we have $S(n) = S(n-1)+S(n-5)$.

Note: Tweaked in appropriate ways, this problem gives interesting sequences. Changing the problem to a *circle* of desks results in the Perrin sequence, which has the neat property whereby if p is prime, then the pth term is divisible by p; the first counterexample to the converse is the 271441th term, only discovered in 1982 by Dan Shanks and Bill Adams (both professors at the University of Maryland). Taking away the requirement that the room be filled, but keeping students separated in a row of desks, results in the Fibonacci sequence. Taking away the requirement that the room be filled, but keeping students separated in a *circle* of desks, results in the Lucas sequence. If the teacher allows at most one (or two or three) students to move to make more room, generalizations of these sequences result.

3. **So primitive, even a caveman could do it.** A Pythagorean triple is a triple of integers (a, b, c) such that $a^2 + b^2 = c^2$. A triple is called *primitive* if and only if the greatest common divisor of its three integers is 1.

   (a) (2 points) Prove that for any positive integers p, q with $p > q$, that $(p^2 - q^2, 2pq, p^2 + q^2)$ is a Pythagorean triple (not necessarily primitive[1]).

   (b) (5 points) Explain why, in a primitive Pythagorean triple, the largest number is always odd, and one of the other numbers is always even.

   (c) (3 points) Define the matrices $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ by

$$\mathcal{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathcal{C} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}.$$

   If we rewrite familiar Pythagorean triples (a, b, c) in matrix form as $[a \quad b \quad c]$, show that the following matrix products are Pythagorean triples:

   i. $[3 \quad 4 \quad 5]\,\mathcal{A}$
   ii. $[3 \quad 4 \quad 5]\,\mathcal{B}$
   iii. $[3 \quad 4 \quad 5]\,\mathcal{C}$, and
   iv. $[5 \quad 12 \quad 13]\,\mathcal{B}$.

   (d) (5 points) If $[a \quad b \quad c]$ is a primitive Pythagorean triple, prove that the matrix product $[a \quad b \quad c]\mathcal{A}$ is also a primitive Pythagorean triple.

[1] A primitive triple will be formed iff p and q have a GCD of one and one of them is even.

Solution:

(a)

$$\begin{aligned} &(p^2 - q^2)^2 + (2pq)^2 \\ &= p^4 - 2p^2q^2 + q^4 + 4p^2q^2 \\ &= p^4 + 2p^2q^2 + q^4 \\ &= (p^2 + q^2)^2 \end{aligned}$$

So by the converse of the Pythagorean Theorem, the triangle is right, hence the (integer) sides are a Pythagorean triple.

(b) If all three numbers in a Pythagorean triple were even, the GCD would be ≥ 2, so by definition the triple would not be primitive.

If all three numbers in a Pythagorean triple were odd, we would have

$$(\text{odd})^2 + (\text{odd})^2 = (\text{odd})^2$$

$$\text{odd} + \text{odd} = \text{odd},$$

which is impossible.

Similarly we can discount

$$(\text{even})^2 + (\text{even})^2 = (\text{odd})^2$$

and

$$(\text{even})^2 + (\text{odd})^2 = (\text{even})^2.$$

This leaves two possibilities:

$$(\text{odd})^2 + (\text{odd})^2 = (\text{even})^2$$

and

$$(\text{even})^2 + (\text{odd})^2 = (\text{odd})^2.$$

The first can be discounted, because any odd number is equivalent to 1 or 3 mod 4; hence any odd number squared is equivalent to 1 mod 4; hence the sum of any two odd numbers, squared, is equivalent to 2 mod 4. By contrast, any even number is equivalent to 0 or 2 mod 4; hence any even number squared is equivalent to 0 mod 4. Since a number equivalent to 2 mod 4 cannot equal a number equivalent to 0 mod 4, the sum of two odd numbers squared cannot equal an even number squared.

This leaves only the last possibility:

$$(\text{even})^2 + (\text{odd})^2 = (\text{odd})^2,$$

e.g., $3^2 + 4^2 = 5^2$.

(c)

$$\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 20 & 29 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 12 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 15 & 8 & 17 \end{bmatrix}$$

$$\begin{bmatrix}5 & 12 & 13\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ -2 & -1 & -2\\ 2 & 2 & 3\end{bmatrix} = \begin{bmatrix}7 & 24 & 25\end{bmatrix}$$

(d) We assume the result provided in the footnote: p and q are coprime integers of opposite parity. So for any primitive triple, we have

$$\begin{bmatrix}(p^2-q^2) & (2pq) & (p^2+q^2)\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 3\end{bmatrix}$$

$$= \begin{bmatrix}(3p^2+4pq+q^2) & (4p^2+2pq) & (5p^2+4pq+q^2)\end{bmatrix}$$
$$= \begin{bmatrix}(4p^2+4pq+q^2-p^2) & (4p^2+2pq) & (4p^2+4pq+q^2+p^2)\end{bmatrix}$$
$$= \begin{bmatrix}((2p+q)^2-p^2) & (2(2p+q)p) & ((2p+q)^2+p^2)\end{bmatrix},$$

the three terms of which are of the form $p'^2 - q'^2, 2p'q', p'^2 + q'^2$, where $p' = 2p + q$ and $q' = p$. By the Euclidean algorithm, $\mathrm{GCD}(2p+q, p) = \mathrm{GCD}(q, p)$, but we know $\mathrm{GCD}(q, p) = 1$. Also, if p is even and q odd, then p is still even while $2p + q$ is odd. Or, if p is odd and q even, p is still odd, and $2p + q$ even. So $2p + q$ and p are indeed coprime integers of opposite parity, so the resulting triple is indeed primitive.

Note: These matrices were first published in 1970 by A. Hall, a teacher at St. John's School, Pinner, Middlesex, England (which soon afterward moved to Northwood, Middlesex). His brief article was called "Genealogy of Pythagorean Triads," in *The Mathematical Gazette,* issue LIV, pp. 377–9.

If $[a \quad b \quad c]$ is a primitive Pythagorean triple, not only is $[a \quad b \quad c]\mathcal{A}$ a primitive Pythagorean triple, but $[a \quad b \quad c]\mathcal{B}$ and $[a \quad b \quad c]\mathcal{C}$ are as well. In fact, every primitive Pythagorean triple can be expressed as $[a \quad b \quad c]\mathcal{X}$, where $\mathcal{X}$ is a (finite) matrix product of $\mathcal{A}$'s, $\mathcal{B}$'s, and $\mathcal{C}$'s.

Hall named his matrices $\mathcal{A}$, $\mathcal{U}$, and $\mathcal{D}$, respectively, which allowed construction of a "genealogical chart," beginning with $(3, 4, 5)$ on the left, three triples in the next column (one Up, one Across, one Down), nine triples in the next column (three connecting to each of the previous three), etc.

Quoting Hall:

> It is interesting to trace certain important lines of descent.
>
> The central line contains all triads in which x and y are consecutive integers.[2] ...
>
> The extreme top line contains all triads in which y and z are consecutive integers, while the extreme lower line contains all in which x and z are consecutive odd numbers.
>
> Further sequences may be traced by alternating branches up, across, and down ($\mathcal{U}$, $\mathcal{A}$ and $\mathcal{D}$):
>
> $\mathcal{UAUA}$... and $\mathcal{AUAU}$... contain triads for which m and n are in the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ...[3] In these triads, x and z are alternately members of the same sequence. ...
>
> $\mathcal{DUDU}$... gives ...triangles in which the hypotenuse and twice the shortest side are consecutive integers, thus tending for form half an equilateral triangle. $\mathcal{UDUD}$... tends to produce the same type of triangle.
>
> ...

[2]Hall is referring to triples x-y-z.

[3]Hall is referring to the triple generator $(m^2 - n^2, 2mn, m^2 + n^2)$

> The difference between x and y in any triad is equal to either the difference or the sum of x and y in the "parent triad." For this reason, the difference between the two shorter sides of any primitive Pythagorean triangle must be either 1 (the difference of 3 and 4) or the sum of the shorter sides in another primitive Pythagorean triangle. The possible differences are therefore: 1, 7, 17, 23, 31, 41, 47, 49, ... It was the investigation of these possible differences, suggested by Mr. P. I. Wyndham, which led to [these] results...

Phrasing Hall's discovery differently, the complete set of primitive Pythagorean triples can be represented as an infinite threefold-branching tree, rooted in $(3, 4, 5)$.

Twelve years later, in the same journal, Alan Wayne of Pasco-Hernando Community College, FL, published an article "A Genealogy of 120° and 60° Natural Triangles," in which he showed that the complete set of integer-sided triangles containing 120° angles can be represented as *two* infinite *five*fold-branching trees, rooted in $(3, 5, 7)$ and $(8, 7, 13)$. Similar to Hall's method, each triple leads to five other triples; and those other triples are found via matrix multiplication.

Wayne also showed that the complete set of integer-sided triangles containing 60° angles (discounting equilateral triangles) can be represented as *four* infinite fivefold-branching trees, rooted in $(3, 8, 7)$, $(8, 15, 13)$, $(5, 8, 7)$, and $(7, 15, 13)$. Again, each triple leads to five other triples; and those other triples are found via matrix multiplication.

4. **Algebra.**

(a) (5 points) Find the sum:

$$\sum_{b=2}^{100} \frac{1}{\log_b 100!}.$$

(b) (10 points) Show that there are infinitely many pairs of positive integers x, y, such that

$$x^{x-y} = y^{x+y}.$$

Solution:

(a) We have

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots\frac{1}{\log_{100} 100!}$$

By the change of base formula,

We have

$$\frac{1}{\frac{\log 100!}{\log 2}}+\frac{1}{\frac{\log 100!}{\log 3}}+\frac{1}{\frac{\log 100!}{\log 4}}+\cdots\frac{1}{\frac{\log 100!}{\log 100}}$$
$$\frac{\log 2}{\log 100!}+\frac{\log 3}{\log 100!}+\frac{\log 4}{\log 100!}+\cdots+\frac{\log 100}{\log 100!}$$
$$\frac{\log 2\cdot 3\cdot 4\cdot\cdots\cdot 100}{\log 100!}$$
$$\frac{\log 100!}{\log 100!}$$
$$\boxed{1}$$

(b) Suppose

$$x^{x-y}=y^{x+y}.$$

Then

$$\frac{x^x}{x^y}=y^xy^y$$
$$\frac{x^x}{y^x}=x^yy^y$$
$$\left(\frac{x}{y}\right)^x=(xy)^y.$$

Since x and y are integers by assumption, the RHS is an integer, so the LHS is, too. This means $\frac{x}{y}$ is an integer z, which means $x=zy$. Substituting, we get

$$z^{zy}=(zyy)^y$$
$$(z^z)^y=(zyy)^y$$

Taking the yth root of both sides,

$$z^z=zy^2$$
$$z^{z-1}=y^2.$$

To find an x, y combination that works, we need a z, y pair to make the above equation work. Let $z=2k+1$.

$$(2k+1)^{2k}=y^2.$$
$$\left((2k+1)^k\right)^2=y^2.$$

Therefore let

$$y=(2k+1)^k,$$
$$x=zy=(2k+1)^{k+1}.$$

Therefore, for any k, if $x=(2k+1)^{k+1}$ and $y=(2k+1)^k$, the equation is true. For example, when $k=1$, $x=9$ and $y=3$. The original equation asserts $9^{9-3}=3^{9+3}$ or $9^6=3^{12}$. If $k=2$, $x=125$ and $y=25$; the original equation asserts $125^{100}=25^{150}$ or $5^{300}=5^{300}$.

Because this works for any k, we have shown that there are infinitely many k that work.

5. **A trigonometric identity.**

(a) (1 point) Prove the identity

$$\cos x = \frac{\sin 2x}{2 \sin x}.$$

(b) (2 points) Prove the identity

$$\cos x + \cos 3x = \frac{\sin 4x}{2 \sin x}.$$

(c) (3 points) Prove the identity

$$\cos x + \cos 3x + \cos 5x = \frac{\sin 6x}{2 \sin x}.$$

(d) (4 points) Prove the identity

$$\cos x + \cos 3x + \cos 5x + \cos 7x = \frac{\sin 8x}{2 \sin x}.$$

(e) (5 points / 15 points) Prove the identity

$$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin(2nx)}{2 \sin x}.$$

(If you are able to prove this identity, you will receive full credit for the problem.)

Solution:

There are multiple ways to solve these problems, but for each problem we show one method.

(a) $\sin 2x = 2\sin x\cos x$ is a well-known identity (from $\sin(a+b) = \sin a\cos b + \cos a\sin b$.) Dividing both sides by $2\sin x$ gives the desired result.

(b) The cosine sum identity gives

$$\begin{aligned}
&\cos x + \cos 3x \\
&= 2\cos\left(\frac{3x-x}{2}\right)\cos\left(\frac{3x+x}{2}\right) \\
&= 2\cos x\cos 2x \\
&= \frac{4\sin x\cos x\cos 2x}{2\sin x} \\
&= \frac{2\sin 2x\cos 2x}{2\sin x} \\
&= \frac{\sin 4x}{2\sin x}
\end{aligned}$$

(c) We have

$$\begin{aligned}
&\cos x + \cos 3x + \cos 5x \\
&= \frac{2\sin x\cos x + 2\sin x\cos 3x + 2\sin x\cos 5x}{2\sin x}
\end{aligned}$$

Using the function-product formula for $\sin\alpha\cos\beta$, we have

$$\begin{aligned}
&= \frac{\frac{2}{2}(\sin 2x + \sin 0) + \frac{2}{2}(\sin 4x + \sin(-2x)) + \frac{2}{2}(\sin 6x + \sin(-4x))}{2\sin x} \\
&= \frac{\sin 2x + \sin 4x - \sin 2x + \sin 6x - \sin 4x}{2\sin x} \\
&= \frac{\sin 6x}{2\sin x}
\end{aligned}$$

(d) Using the cosine sum identity, we have

$$\begin{aligned}
&\cos x + \cos 7x + \cos 3x + \cos 5x \\
&= 2\cos 4x\cos 3x + 2\cos 4x\cos x \\
&= 2\cos 4x(\cos 3x + \cos x)
\end{aligned}$$

Using the result from (b), this equals

$$\begin{aligned}
&= 2\cos 4x\left(\frac{\sin 4x}{2\sin x}\right) \\
&= \frac{\sin 8x}{2\sin x}
\end{aligned}$$

(e) We have shown in (a) through (d) that the identity works for $n = 1$ to 4.

Assume the identity works for a certain n. We want to show that it works for $n+1$.

$$\sum_{k=1}^{n+1}\cos(2k-1)x = \sum_{k=1}^{n}\cos(2k-1)x + \cos(2n+1)x.$$

By the inductive hypothesis,

$$\sum_{k=1}^{n+1} \cos(2k-1)x = \frac{\sin(2nx)}{2\sin x} + \cos(2n+1)x$$

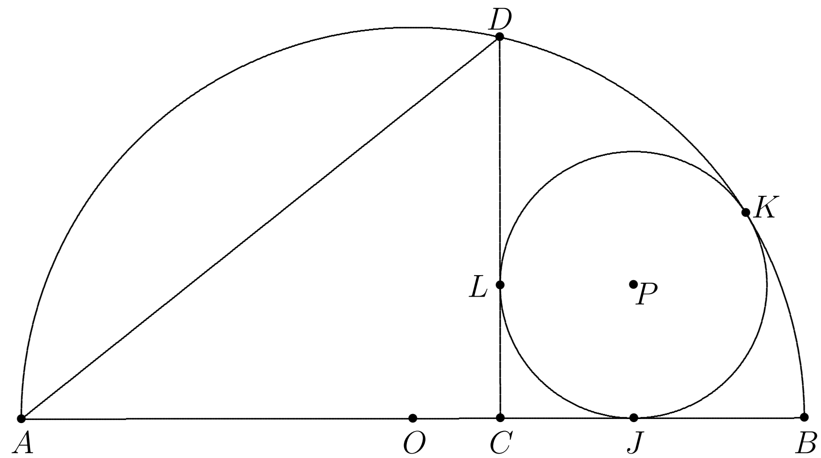
$$= \frac{\sin(2nx)}{2\sin x} + \frac{2\sin(x)\cos(2n+1)x}{2\sin(x)}$$

$$= \frac{\sin(2nx)}{2\sin x} + \frac{\sin((2n+2)x) - \sin(2nx)}{2\sin(x)}$$

$$= \frac{\sin((2n+2)x)}{2\sin(x)}$$

$$= \frac{\sin(2(n+1)x)}{2\sin x}.$$

6. **Geometry** (10 points) In the sketch below, $\overline{CD}$ is perpendicular to the diameter $\overline{AB}$ of the semicircle with center O. The inscribed circle with center P is tangent to $\overline{AB}$ at J, $\overline{CD}$ at L, and the semicircle at K. Show that the line segments $\overline{AD}$ and $\overline{AJ}$ have the same length.



Solution:

(a) Draw $\overline{OK}$, noticing that P lies on $\overline{OK}$ since a line through K perpendicular to the common tangent line must pass through the center of both circles. Let the radius of the semicircle be R and the radius of the inscribed circle be r, making $OP = OK - KP = R - r$. Also, let $AC = x$ and $CB = y$. Without loss of generality, assume $x > R > y$, as drawn.

By the Pythagorean theorem, we have

$$OJ^2 + JP^2 = OP^2$$

$$(OC + CJ)^2 + JP^2 = OP^2$$
$$((x - R) + r)^2 + r^2 = (R - r)^2$$
$$x^2 + R^2 + r^2 + 2xr - 2xR - 2rR + r^2 = R^2 - 2rR + r^2.$$

Cancelling, we have

$$x^2 + r^2 + 2xr - 2xR = 0$$
$$(x + r)^2 - 2xR = 0$$
$$(x + r)^2 = 2xR$$
$$(x + r)^2 = 2x\frac{x + y}{2}$$
$$(x + r)^2 = x(x + y).$$

Therefore $AJ = x + r = \sqrt{x(x + y)} = \sqrt{x^2 + xy}$.

By similarity of triangles, CD is the geometric mean of x and y.

Again by the Pythagorean Theorem,

$$AD^2 = AC^2 + CD^2$$

$$AD^2 = x^2 + (\sqrt{xy})^2$$
$$AD^2 = x^2 + xy$$

Therefore $AD = \sqrt{x^2 + xy}$.

Since both AD and AJ equal $\sqrt{x^2 + xy}$, we have $AD = AJ$ as required.

7. **The absent-minded mathematician.**

(a) (1 point) A busy math teacher writes college recommendations for six students, each applying early-admission to a different college. In his haste to get to class, he stuffs each letter into one of the preaddressed envelopes and seals them. What is the probability that each letter is in the correct envelope?

(b) (4 points) What is the probability that *none* of the letters is in the correct envelope?

(c) (10 points) The same teacher has five pairs of socks, each a different shade of gray. He is too busy to sort his socks, so every Sunday after doing his laundry he randomly pairs up the ten socks, creating five pairs. Then every day from Monday to Friday he wears a different pair, so all five pairs are used during the week.

When the teacher wears socks that are either the same shade or differ by one shade, his students don't notice anything wrong. But when his socks differ by more than one shade, his students laugh at him. What is the probability that he makes it through the week without getting laughed at?

Solution:

(a) There are 6! ways to arrange the six letters, and only one arrangement is correct, so the answer is $1/6! = \boxed{1/720.}$

(b) This is just a standard derangement problem. There are six envelopes. The number of ways that a specific letter is in its correct envelope is 5! (because the other five letters can be rearranged), and there are $\binom{6}{1}$ ways to choose the one letter. The number of ways that two specific letters are in their correct envelopes is 4! (because the other four can be rearranged), and there are $\binom{6}{2}$ ways to choose two letters. And so on.

Using inclusion-exclusion, the number of ways that at least one letter is in the right place is

$$\binom{6}{1} \cdot 5! - \binom{6}{2} \cdot 4! + \binom{6}{3} \cdot 3! - \binom{6}{4} \cdot 2! + \binom{6}{5} \cdot 1! - \binom{6}{6} \cdot 0!$$

$$6 \cdot 120 - 15 \cdot 24 + 20 \cdot 6 - 15 \cdot 2 + 6 \cdot 1 - 1 \cdot 1$$

$$720 - 360 + 120 - 30 + 6 - 1$$

$$455$$

If there are 455 ways to get at least one envelope right, there are $6! - 455 = 265$ ways to get them all wrong. Hence the probability is

$$\frac{265}{6!} = \frac{265}{720} = \boxed{\frac{53}{144}.}$$

(c) Both solution methods detailed below require one key observation: there are very few ways that the math teacher can pair his socks without being laughed at. If we label the socks A, B, C, D, and E by their colors (two of each), then sock A can be "fashionably" paired with the other sock A or with one of the two socks B. But if one sock A is paired with sock B, the remaining socks are

$$\{A, B, C, C, D, D, E, E\},$$

meaning that the other sock A must pair with the other sock B if the teacher is to avoid being laughed at.

The same occurs in the middle of the color gradient: If a sock C is paired with a sock D, the other sock C must be paired with the other sock D. For if the other sock C were paired with a sock B, the remaining socks would be

$$\{A, A, B, D, E, E\},$$

leaving three socks on either side to pair up, a parity problem.

With this observation made, there are fundamentally only eight ways the socks can be paired. Mismatched pairs are shown in boldface:

AA	BB	CC	DD	EE	I. All matched pairs
AB	**AB**	CC	DD	EE	II. Two fashionable pairs, three matched pairs
AA	**BC**	**BC**	DD	EE	
AA	BB	**CD**	**CD**	EE	
AA	BB	CC	**DE**	**DE**	
AB	**AB**	**CD**	**CD**	EE	III. Four fashionable pairs, one matched pair
AB	**AB**	CC	**DE**	**DE**	
AA	**BC**	**BC**	**DE**	**DE**	

Method One: For this solution method, we consider the socks distinguishable, and count possible pairings of socks. There are

$$\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{10!}{2^5} = 113400$$

ways to choose five groups of two. To repeat, this assumes socks are distinguishable: if they were not distinguishable, this would overcount.

Referring to the table on the previous page,

I. A "Type I" arrangement can be created $5! = 120$ ways by rearranging the groups. (We need not account for the distinguishability of two matching socks, because a group containing A_1 and A_2 is the same as a group containing A_2 and A_1.)

II. Each "Type II" arrangement can be selected $5! \cdot 2 = 240$ ways. e.g., for the second row, the two AB entries are distinguishable, giving 5! arrangements. We multiply by 2 because A_1 might be paired with either B_1 or B_2.

III. Each "Type III" arrangement can be selected $5! \cdot 2^2 = 480$ ways. e.g., for the sixth row, the AB and CD entries are distinguishable, giving 5! arrangements. We multiply by 2 because A_1 might be paired with either B_1 or B_2, and again by 2 because C_1 might be paired with either D_1 or D_2.

This gives a numerator of $120 + 4 \cdot 240 + 3 \cdot 480 = 2520$ pairings that are fashionable. So the teacher's chances are

$$\frac{2520}{113400} = \boxed{\frac{1}{45}.}$$

Method Two: For this solution method, we consider the socks indistinguishable, and simply count strings. We assume that the teacher puts the ten socks in random order, then makes a pair from the first two, second two, etc.

By basic combinatorics, the number of arrangements of the letters AABBCCDDEE is

$$\frac{10!}{2!\,2!\,2!\,2!\,2!} = 113400.$$

Referring to the table on the previous page,

I. There are $5! = 120$ "Type I" arrangements (corresponding to the five places for each correct pair of socks).

II. Each "Type II" possibility has 240 arrangements. e.g., for the second row, the string AB-AB-CC-DD-EE has $\frac{5!}{2!}$ arrangements, as does the string BA-BA-CC-DD-EE. The string AB-BA-CC-DD-EE has 5! arrangements because all five two-letter strings are distinguishable. This leads to $60 + 60 + 120 = 240$ strings.

III. Each "Type III" possibility has 480 arrangements. e.g., for the sixth row, the strings AB-AB-CD-CD-EE, BA-BA-CD-CD-EE, AB-AB-DC-DC-EE, and BA-BA-DC-DC-EE all have $\frac{5!}{2!2!} = 30$ arrangements. The strings AB-BA-CD-CD-EE, AB-BA-DC-DC-EE, AB-AB-CD-DC-EE, and BA-BA-CD-DC-EE all have $\frac{5!}{2!} = 60$ arrangements. Finally, the string AB-BA-CD-DC-EE has $5! = 120$ arrangements. This leads to a total of $120 + 4 \cdot 240 + 3 \cdot 480 = 2520$ strings corresponding to acceptable socks for a week. So the teacher's probability of a non-embarassing week is

$$\frac{2520}{113400} = \boxed{\frac{1}{45}.}$$

Note: Questions 4, 6, and 7(c) are from the Walker Prize Examination at Amherst College.

The 2007 Massachusetts Mathematics Olympiad and these solutions were both typeset in Computer Modern using the mathematical typesetting package LaTeX.

Graphics were created using WinGeom, programmed by Rick Parris of Phillips Exeter Academy, and available at `http://math.exeter.edu/rparris/wingeom.html`. WinGeom can export graphics in PiCTeX format, allowing the graphics to be part of the LaTeX source file (and to look very nice).

Special thanks to Harold Reiter for providing examples of LaTeXed questions from past AMC contests, which made the typesetting task much easier.