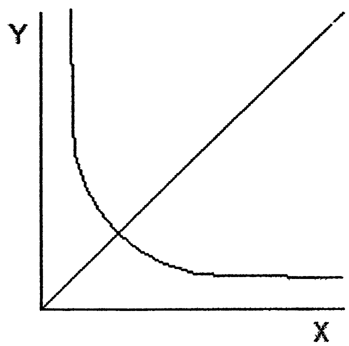
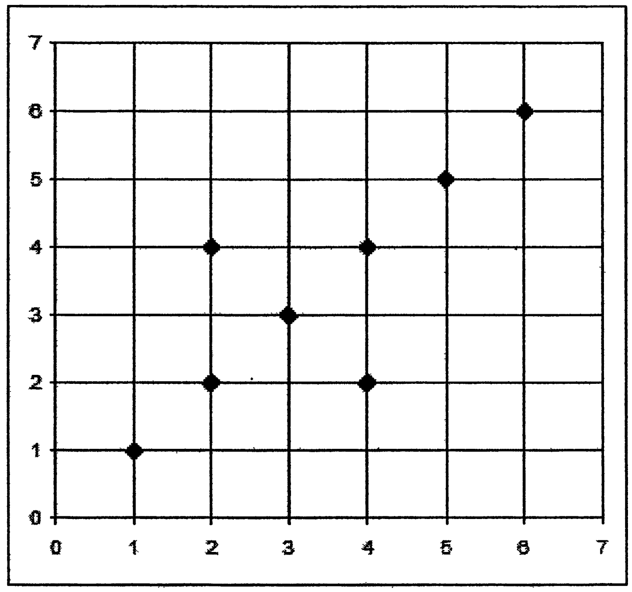
SOLUTIONS – OLYMPIAD EXAMINATION LEVEL 2

1. a) If x and y are positive integers less than 7, sketch the graph of the solution set to the equation $x^y = y^x$.



Continuous Solution Set



Discrete Solution Set

b) If $x^2 + 9x + 20$ and $2x + 10$ are both positive integers, determine all real solutions to $\left(x^2 + 9x + 20\right)^{(2x+10)} = (2x + 10)^{(x^2+9x+20)}$.

This problem is in the form $x^y = y^x$. Check both y=x and the separate cases x=2,y=4 , and x=4,y=2.

$x^2 + 9x + 20 = 2x + 10$

$x = -2, -5$

Also check $2^4=4^2$ $x^2 + 9x + 20 = 2$ $\quad$ $2x + 10 = 4$

$x = -3, -6$ $\quad$ $x = -3$

This x=-3 is indeed a root.

We can also check it the other way.

$x^2 + 9x + 20 = 4$ $\quad$ $2x + 10 = 2$

$x = irrational$ $\quad$ $x = -4$

These x values don't agree.

We have x = -2,-5,-3. Plugging these in gives $6^6=6^6$, $0^0=0^0$, $2^4=4^2$, so we find that -5 is extraneous. Our result is x=-2,-3

c) If $x^2 - 11x + 30$ and $x^2 - 9x + 18$ are both positive integers, solve
$$\left(x^2 - 11x + 30\right)^{(x^2-9x+18)} = \left(x^2 - 9x + 18\right)^{(x^2-11x+30)}.$$

Same process as above. The equation looks like $x^y = y^x$.

$y = x$ case yields $x^2 - 11x + 30 = x^2 - 9x + 18 \rightarrow x = 6$.

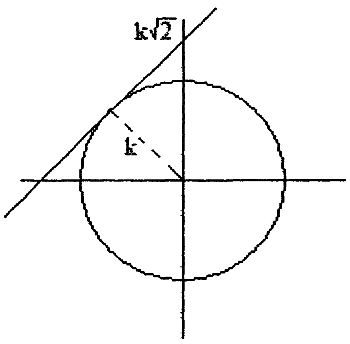
(2,4) case yields $\begin{matrix} x^2 - 11x + 30 = 2 \\ x = 4,7 \end{matrix}$ $\quad \begin{matrix} x^2 - 9x + 18 = 4 \\ x = 2,7 \end{matrix} \rightarrow x = 7$.

(4,2) case yields $\begin{matrix} x^2 - 11x + 30 = 4 \\ x = \dfrac{11 \pm \sqrt{17}}{2} \end{matrix}$ $\quad \begin{matrix} x^2 - 9x + 18 = 2 \\ x = \dfrac{9 \pm \sqrt{17}}{2} \end{matrix}$ $\rightarrow$ no solutions.

Check solutions : $x = 6$ gives $0^0 = 0^0$. $x = 7$ gives $2^4 = 4^2$.

$x = 7$ is the only solution.

2. a) Let S be the set of circles $x^2 + y^2 = n$ for $n = 1, 4,$ and 9. Let T be the set of lines $y = x + k\sqrt{2}$ for $k = 0, \pm 1, \pm 2, \pm 3,$ and ± 4. Determine the number of points in the intersection of the graphs of the sets S and T.



The line $y = x + k\sqrt{2}$ will intersect circles with radius greater than $|k|$ twice, circles with radius $|k|$ once, and circles with radius less than $|k|$ never, as shown in the diagram.

$n = 1, 4,$ and 9 correspond to $r = 1, 2,$ and 3.

For $r=1$, the intersections will be 2 with $k = 0$, 1 with $k = 1$, and 1 with $k = -1$, a total of 4 intersections.
For $r=2$, there's 2 intersections with $k=-1,0,1$, and 1 intersection with each k=-2,2, for a total of 8 intersections.
For $r=3$, there's 2 intersections with k=-2,-1,0,1,2, and 1 intersection with each k=-3,3, a total of 12 intersections.
So there are 4 + 8 + 12 = 24 intersections.

b) Let S be the set of circles $x^2 + y^2 = n$ for $n = 1, 2, 3, \ldots, 8, 9$. Let T be the set of lines $y = x + k\sqrt{2}$ for all integers k. Determine the number of points in the intersection of the graphs of the sets S and T.

We already solved this for radii 1,2, and 3. The addition of "all integers k" wont change anything because all lines with $|k| > 3$ have 0 intersections with these circles.
Radius = $\sqrt{2}, \sqrt{3}$ hit by k=1 twice, k=0 twice, and k=-1 twice. = 6
Radius = $\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$ each have 10 intersections.
Total intersections = $4 + 2 \cdot 6 + 8 + 4 \cdot 10 + 12$ = 76 intersections

c) Let p be an arbitrary positive integer greater than 1. Let S be the set of circles $x^2 + y^2 = n$ where n takes on all positive integer values less than p^2. Let T be the set of lines $y = x + k\sqrt{2}$ for all integers k. Determine, in terms of p, the number of points in the intersection of the graphs of the sets S and T.

The problem asks to find the number of intersections with circles of radii less than p (the radii are all square roots of integers).
Using the data we've already established for radii up to 3 and then continuing on by the same process, we can make a table listing the number of intersections:

$1 \le r < 2 \ldots 4 + 2 \cdot 6$

$2 \le r < 3 \cdots 8 + 4 \cdot 10$

$3 \le r < 4 \cdots 12 + 6 \cdot 14$

$n \le r < n+1 \cdots 4n + 2n \cdot (4n+2) = 8n^2 + 8n$
These patterns will always hold because the number of intersections will grow linearly as well as the number of integers between 2 perfect squares.

To find the number of intersections for radii less than p, we take p-1 of these terms.

$$\sum_{n=1}^{p-1} 8n^2 + 8n = 8\sum_{n=1}^{p-1} n^2 + 8\sum_{n=1}^{p-1} n = 8 \cdot \frac{1}{6} \cdot (p-1)(p)(2(p-1)+1) + 8 \cdot \frac{(p-1)(p)}{2}$$

Answer $= \frac{8}{3}\left(p^3 - p\right)$

Note: The sum of the first n squares is given here as a simple cubic formula. Students should know that it is a cubic polynomial that is derived by induction. If the formula can not be remembered or re-derived by the student, partial credit will be given for the answer in summation notation.

3. a) Given the following sequence, $a_1 = 1$ and $a_n = 1 + 2a_{n-1}$ for $n > 1$, find a formula for the nth term explicitly in terms of n. Prove that your formula is correct using an inductive proof.

After writing a few terms (1, 3, 7, 15), it is apparent that $a_n = 2^n - 1$.

Inductive proof:

Claim $a_n = 2^n - 1$ for all positive integers n.

Base case: $a_1 = 1 = 2^1 - 1$

Inductive step: Assume $a_k = 2^k - 1$.

$a_{k+1} = 1 + 2a_k = 1 + 2\left(2^k - 1\right) = 2 \cdot 2^k - 1 = 2^{k+1} - 1$

$a_{k+1} = 2^{k+1} - 1$

Therefore $a_n = 2^n - 1$ for all positive integers n by mathematical induction.

b) Given the following sequence, $a_1 = 7$, $a_2 = 9$, and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$,

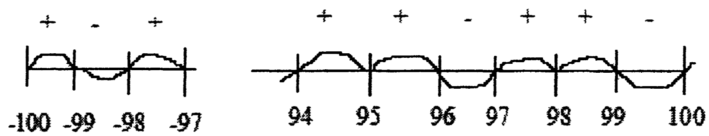
i) Write out the first 8 terms.

ii) Consider $f(x) = (x - 100)^{a_1}(x - 99)^{a_2}(x - 98)^{a_3} \mathrm{K}\ (x + 99)^{a_{200}}(x + 100)^{a_{201}}$. If a number x is drawn at random from the interval (–100, 100), determine the probability that $f(x)$ is positive.

i) 7, 9, 16, 25, 41, 66, 107, 173

ii) We have a polynomial written in factored form. First we need to determine the parity of the degree (the degree written modulo 2, or the remainder when the degree is divided by 2). The degree of the polynomial is the sum of the first 201 terms of the sequence. The terms written modulo 2 are 1,1, 1+1 = 0, 1+0 =1, 0+1=1, 1+1=0. By this point we see repetition; 1,1,0,1,1,0,1,1,0... Now if we compute S_n (the sum of the first n terms) also in modulo 2, the pattern is 1,0,0,1,0,0,1,0,0... so after 201 terms the sum will be even, so the degree of the polynomial is even. This means that for values of x less than -100 or greater than 100, f(x) will be positive.

If a root has an even exponent, then the signs of f(x) will be the same on both sides of the root. If a root has an odd exponent, then the signs of f(x) will be different. We can use this to construct a rough graph.



I started with the x=100 side because the (x-100) term with a_1 corresponds to the root x=100. A pattern of { -, +, +} develops, moving leftward. The pattern occurs 66 times between x=-98 and x=100. That means 132 positive sections and then one more positive section on $-100<x<-99$. The probability that f(x) will be positive is **133/200**.

c) Let a_n be the nth positive integer with an even number of positive factors.
Note that 1 has 1 factor and 8 has 4 factors, 1, 2, 4, and 8.

i) Write down the first twelve members of the sequence using $a_1 =\ \ $, $a_2 =\ \ $, etc.

ii) Find an explicit formula for a_n in terms of n. Explain how you arrived at the formula. You may wish to use the greatest integer function in your answer.

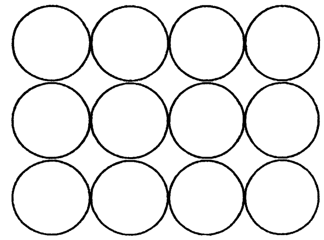
i) $a_1 = 2, a_2 = 3, a_3 = 5, a_4 = 6, a_5 = 7, a_6 = 8,$
$a_7 = 10, a_8 = 11, a_9 = 12, a_{10} = 13, a_{11} = 14, a_{12} = 15$

ii) The hard part of the problem is when given n, to find out which 2 perfect squares a_n lies between, for example if a_n is between 4 and 9, then the easy solution is $a_n = n+2$. If a_n is between 9 and 16, $a_n = n+3$. The idea is to create some function that will tell us which squares we're in between. If n=1, we're in the 0th group, if n=3, we're in the 1st group, if n=7, we're in the 2nd group, etc. g(1) = 0, g(3) = 1, g(7) = 2, g(13) = 3. I set the first number in each group to an integer so that g(x) rounded down will give the group number, for example g(8) will be near 2.1 or so and get rounded down to show it is in the 2nd group. The above pattern is g(1 + (k+1)(k)) = k, or $g^{-1}(k) = k^2 + k + 1$. Using the inverse trick in algebra 2 (if y = 3x + 1, to find the inverse solve x = 3y + 1 for y), you get $g(k) = \dfrac{-1+\sqrt{4n-3}}{2}$. So the "group" of k is [g(k)]. And we see $a_n = n + [g(n)] + 1$.

Answer: $a_n = n + \left\lfloor \dfrac{-1+\sqrt{4n-3}}{2} \right\rfloor + 1$

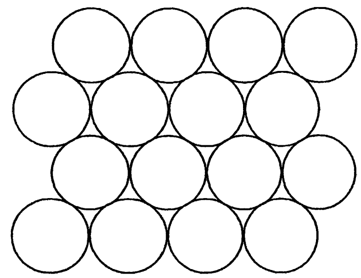
4.

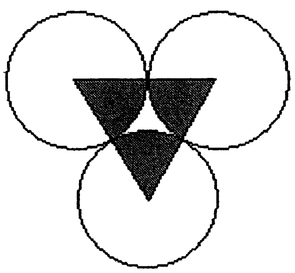
a) To the nearest tenth what percent of a plane is filled by congruent circles if they are arranged in a rectangular grid as shown?



Draw a circle of radius 1 inside a square of side length 2. The circle has area π and the square has area 4. This pattern repeats infinitely many times in this rectangular grid. The area fraction is $\frac{\pi}{4} \approx 78.6\%$.

b) To the nearest tenth what percent of a plane is filled by congruent circles if they are arranged in a hexagonal grid as shown?





Consider a unit cell for this diagram. That is, a unit that when repeated over and over again generates exactly this grid. A unit cell for this diagram can be found by drawing an equilateral triangle connecting the centers of 3 adjacent circles. We will assume for ease of calculation that the radii are all 1 unit, as answer is not dependant on this length. The shaded region is 3 60 degree sectors, which have a total area of $\frac{\pi}{2}$. The area of the entire equilateral triangle is $\frac{2^2\sqrt{3}}{4} = \sqrt{3}$. The resulting fraction of filled area is $\frac{\pi}{2\sqrt{3}} = \frac{\pi\sqrt{3}}{6} \approx 90.7\%$.

c) To the nearest tenth what percent of space is filled by congruent spheres if they are arranged in a rectangular grid? As with circles in a rectangular grid the lowest point of each sphere is tangent to the highest point of the sphere directly below it.

Draw a sphere of radius 1 inside a cube of side length 2. This is the unit cell for the 3-dimensinal rectangular grid. The sphere has a volume of $\frac{4\pi}{3}$. The cube has a volume of 8. The resulting fraction of filled space is $\frac{\pi}{6} \approx 52.4\%$.

d) To the nearest tenth what percent of space is filled by congruent spheres if they are arranged in a hexagonal grid? Analogous to circles in a hexagonal grid, each sphere nestles down in the space formed by the three spheres right below it.

The unit cell approach is possible for this problem but I use an easier method here. Let the spheres all have radii 1. If a top view of one layer is drawn, we see circles in the 2-dimensional hexagonal grid (part b). The horizontal distance between circles is 2. The vertical distance between rows is $\sqrt{3}$. When another layer of spheres is added on the top of this, the centers of 4 adjacent spheres create a regular tetrahedron. The distance in the "z direction" between 2 of these layers is just the height of a regular tetrahedron with side length 2, $\frac{2\sqrt{6}}{3}$. One sphere of volume $\frac{4\pi}{3}$ takes up a space in the box of $2 \cdot \sqrt{3} \cdot \frac{2\sqrt{6}}{3} = 4\sqrt{2}$. This is found just by taking the product of the distance to the next row of spheres along each of the 3 axes. The fraction of filled space is then $\frac{4\pi}{3} \cdot \frac{1}{4\sqrt{2}} = \frac{\pi\sqrt{2}}{6} \approx 74.0\%$.

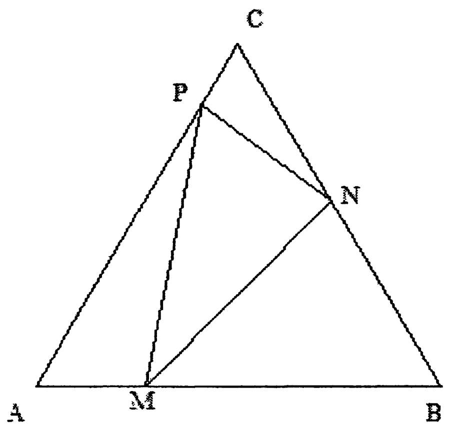
5. a) For $x, y > 0$, prove that $\frac{x^2}{4y^2} + \frac{y^2}{x^2} \geq 1$.

$$\left(x^2 - 2y^2\right)^2 \geq 0$$

$$x^4 + 4y^4 \geq 4x^2y^2$$

$$\frac{x^2}{4y^2} + \frac{y^2}{x^2} \geq 1$$

b) Triangle ABC is equilateral. Points M, N, and P lie on $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively. Prove: $AM \cdot AP + BM \cdot BN + CN \cdot CP < (AB)^2$.



Draw a triangle connecting M, N, and P.
Area ΔAMP + Area ΔBMN + Area ΔCNP < Area ΔABC from the picture.

Use Area = $\frac{ab \cdot \sin(C)}{2}$ on the 4 triangles in the inequality.

$$\frac{AM \cdot AP \cdot \sin(60^o)}{2} + \frac{BM \cdot BN \cdot \sin(60^o)}{2} + \frac{CN \cdot CP \cdot \sin(60^o)}{2} < \frac{AB \cdot AC \cdot \sin(60^o)}{2}$$

$$\frac{\sqrt{3}}{4}\left(AM \cdot AP + BM \cdot BN + CN \cdot CP\right) < \frac{\sqrt{3}}{4} AB^2$$

$$AM \cdot AP + BM \cdot BN + CN \cdot CP < AB^2$$

6. The length of a repeating decimal is the number of digits in each repetition. For example, length of the repeating decimal $\frac{7}{54} = .1296\overline{296}$ is 3 and the length of $.2717\overline{2717}$ is 4.

a) Let n be a positive integer. Explain why the maximum length of a repeating decimal of the form $\frac{1}{n}$ is $n-1$.

When doing long division $\frac{1}{n}$, there is a remainder after every digit. When this remainder repeats to a previous value, the following decimal digits must also repeat because this remainder is the only value that determines the next digits. There are n-1 possible remainders (1 to n-1, 0 is not possible because that would be a terminating decimal, not a repeating decimal), so the maximum length of a repeating decimal of the form $\frac{1}{n}$ is n-1.

b) Find all prime numbers p such that their reciprocals have repeating decimals of length 3 or less. Prove that your list is exhaustive.

Let $\frac{1}{p}$ be a repeating decimal of length L. It can be written:

$$\frac{1}{p} = X \cdot 10^{-L} + X \cdot 10^{-2L} + X \cdot 10^{-3L} \cdots$$

X is an L digit integer in this equation, or less than L digits with some leading zeroes (example X=009 would be ok for L=3).

$\frac{1}{p} = X\left(10^{-L} + 10^{-2L} + 10^{-3L} \cdots\right)$, now $\frac{1}{p}$ can be rewritten as the sum of an infinite geometric series.

$$\frac{1}{p} = X\left(\frac{10^{-L}}{1-10^{-L}}\right) \rightarrow Xp = \frac{1-10^{-L}}{10^{-L}} = 10^{L} - 1$$

Take the L=1, L=2, and L=3 cases we're interested in:

L=1 $\rightarrow Xp = 10^1 - 1 = 9 = 3 \cdot 3$

L=2 $\rightarrow Xp = 99 = 3 \cdot 3 \cdot 11$

L=3 $\rightarrow Xp = 999 = 3 \cdot 3 \cdot 3 \cdot 37$

So the only primes p that solve the problem are p=*3, 11, 37*. There are no other prime factors in the products Xp for L=1,2, or 3. 3 possible solutions are:

X=3, p=3

X=09, p=11

X=027, p=37

c) Let p be any prime greater than 3. Prove that the sum of the digits in one block of the repeating digits of $\frac{1}{p}$ is divisible by 9.

Use the same representation of a repeating decimal as in part (b):

$$Xp = 10^L - 1 = 99999\cdots$$

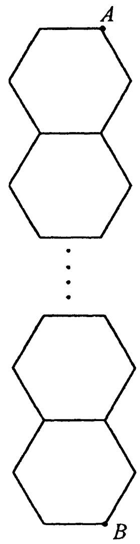
$$X = \frac{9(11111\cdots)}{p}$$

p cant divide 9 because p is prime (and $p > 3$), so X is a multiple of 9.

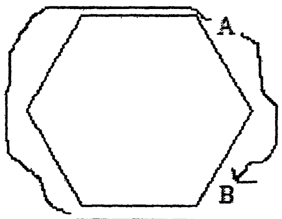
By the rule of 9's, the sum of X's digits must also be a multiple of 9.

7. In problems (a) and (b), the problem is to determine the number of distinct paths from A to B along hexagonal grids subject to the following rules: 1) the path can never move upward and 2) the path can never retrace a segment.

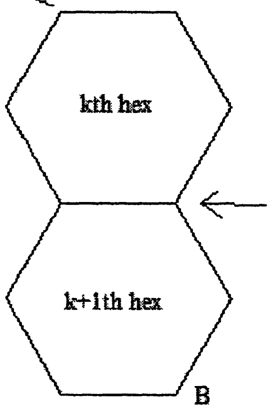
a) This hexagonal grid consists of a column of n hexagons stacked one on top of another as shown. Determine the number of paths from A to B in terms of n. Explain and justify your answer.



The answer is that there are 2^n possible paths from A to B.

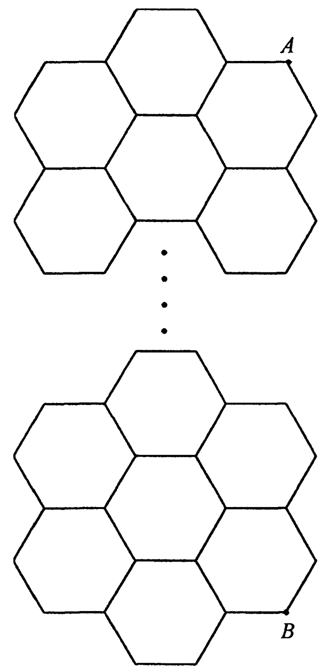


For n=1, there are 2 possible paths.

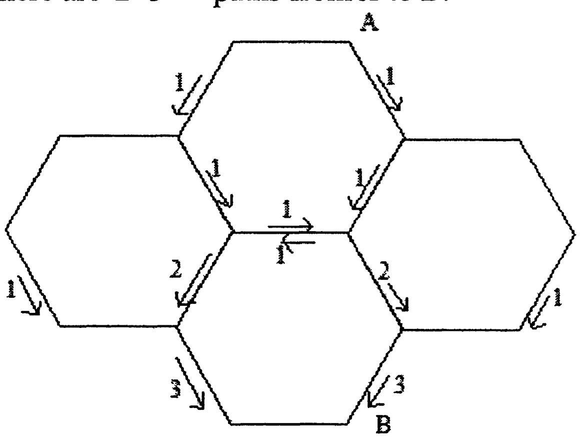


Assume that there are 2^k paths to the level of the bottom of the kth hexagon. Each path can then take either the left or right route to get to point B at the bottom of the k+1th hexagon. The number of possible paths doubles at an intersection like this. Thus, there are 2^{k+1} paths to the bottom of the k+1th hexagon. Apply induction to this process and you arrive at 2^n possible paths from A to B.

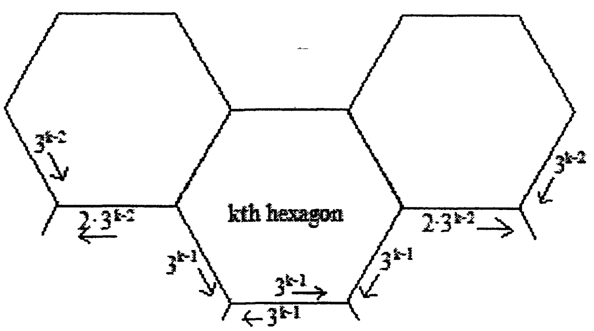
b) This hexagonal grid consists of 3 columns of $n - 1$, n, and $n - 1$ hexagons respectively as shown. Determine the number of paths from A to B in terms of n. Explain and justify your answer.



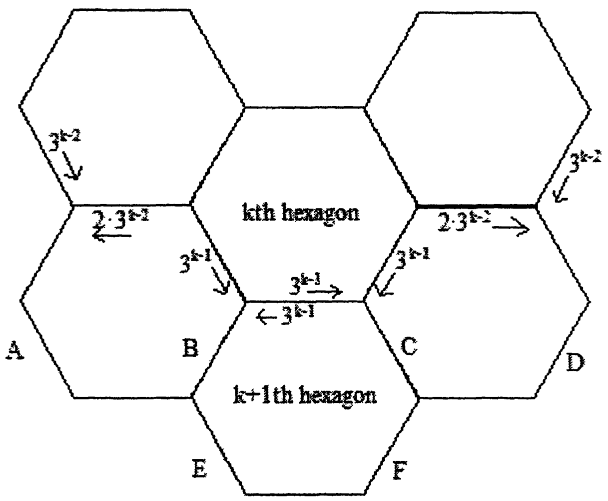
There are $2 \cdot 3^{n-1}$ paths from A to B.



There are 2 paths for the trivial n=1 case again, and 6 paths for the n=2 case, as shown in the diagram.



We assume these values of number of paths in each direction for the kth hexagon (which are all valid for the n=2 case above). Then we will add the k+1th row of 3 hexagons below and compute the number of paths again.



$A = 3^{k-2} + 2 \cdot 3^{k-2} = 3^{k-1}$

$B = 3^{k-1} + 3^{k-1} = 2 \cdot 3^{k-1}$

$C = 3^{k-1} + 3^{k-1} = 2 \cdot 3^{k-1}$

$D = 3^{k-2} + 2 \cdot 3^{k-2} = 3^{k-1}$

$E = A + B = 3^k$

$F = C + D = 3^k$

paths to bottom $= E + F = 2 \cdot 3^k$

All of the path numbers simply multiply by 3 from the kth row to the k+1th row. By induction, the number of paths from A to B is $2 \cdot 3^{n-1}$.