The Forty-Second Annual

Mathematics Olympiad Competition

Second Level

Friday, March 3, 2006

Sponsored by

The Massachusetts Association of

Mathematics Leagues

and

The Actuaries′ Club of Boston

OLYMPIAD EXAMINATION LEVEL 2

1. a) If x and y are positive integers less than 7, sketch the graph of the solution set to the equation $x^y = y^x$. (2 points)

b) If $x^2 + 9x + 20$ and $2x + 10$ are both positive integers, determine all solutions to $\left(x^2 + 9x + 20\right)^{(2x+10)} = (2x + 10)^{(x^2+9x+20)}$. (5 points)

c) If $x^2 - 11x + 30$ and $x^2 - 9x + 18$ are both positive integers, determine all integer solutions to $\left(x^2 - 11x + 30\right)^{(x^2-9x+18)} = \left(x^2 - 9x + 18\right)^{(x^2-11x+30)}$.

(5 points)

2. a) Let S be the set of circles $x^2 + y^2 = n$ for $n = 1, 4$, and 9. Let T be the set of lines $y = x + k\sqrt{2}$ for $k = 0, \pm 1, \pm 2, \pm 3$, and ± 4. Determine the number of points in the intersection of the graphs of the sets S and T. (3 points)

b) Let S be the set of circles $x^2 + y^2 = n$ for $n = 1, 2, 3, \ldots, 8, 9$. Let T be the set of lines $y = x + k\sqrt{2}$ for all integers k. Determine the number of points in the intersection of the graphs of the sets S and T. (5 points)

c) Let p be an arbitrary positive integer greater than 1. Let S be the set of circles $x^2 + y^2 = n$ where n takes on all positive integer values less than p^2. Let T be the set of lines $y = x + k\sqrt{2}$ for all integers k. Determine, in terms of p, the number of points in the intersection of the graphs of the sets S and T. (6 points)

3. a) Given the following sequence, $a_1 = 1$ and $a_n = 1 + 2a_{n-1}$ for $n > 1$, find a formula for the *n*th term explicitly in terms of *n*. Prove that your formula is correct using an inductive proof. (4 points)

b) Given the following sequence, $a_1 = 7$, $a_2 = 9$, and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$,

i) Write out the first 8 terms. (1 point)

ii) Consider $f(x) = (x-100)^{a_1}(x-99)^{a_2}(x-98)^{a_3} \ldots\ldots (x+99)^{a_{200}}(x+100)^{a_{201}}$.

If a number *x* is drawn at random from the interval $-100<x<100$, determine the probability that $f(x)$ is positive. (5 points)

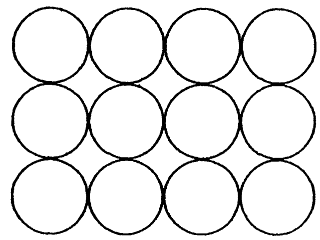
c) Let a_n be the *n*th positive integer with an even number of positive factors. Note that 1 has 1 factor and 8 has 4 factors : 1, 2, 4, and 8.

i) Write down the first twelve members of the sequence using $a_1 =$, $a_2 =$, etc.
(2 points)

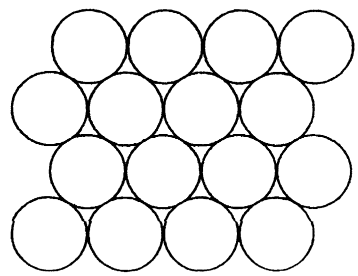
ii) Find an explicit formula for a_n in terms of *n*. Explain how you arrived at the formula. You may wish to use the greatest integer function in your answer.
(5 points)

4. These problems deal with the amount of waste that different ways of packing circles and spheres incur. In each problem assume that the enclosing plane or box is infinitely large. That is, ignore any waste that arises from the edges since there are no edges.

a) To the nearest tenth what percent of a plane is filled by congruent circles if they are arranged in a rectangular grid as shown? (2 points)



b) To the nearest tenth what percent of a plane is filled by congruent circles if they are arranged in a hexagonal grid as shown? (4 points)



c) To the nearest tenth what percent of space is filled by congruent spheres if they are arranged in a rectangular grid? As with circles in a rectangular grid the lowest point of each sphere is tangent to the highest point of the sphere directly below it.

(3 points)

d) To the nearest tenth what percent of space is filled by congruent spheres if they are arranged in a hexagonal grid? Analogous to circles in a hexagonal grid, each sphere nestles down in the space formed by the three spheres right below it.

(6 points)

5. a) For $x, y > 0$, prove that $\frac{x^2}{4y^2} + \frac{y^2}{x^2} \geq 1$. (6 points)

b) Triangle ABC is equilateral. Points M, N, and P lie on $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively, and none of the points M, N, or P coincide with any of the vertices A, B, or C.

Prove: $AM \cdot AP + BM \cdot BN + CN \cdot CP < (AB)^2$. (7 points)

6. The length of a repeating decimal is the number of digits in each repetition. For example, length of the repeating decimal $\frac{7}{54} = .1\overline{296}\overline{296}$ is 3 and the length of $.\overline{2717}\overline{2717}$ is 4.

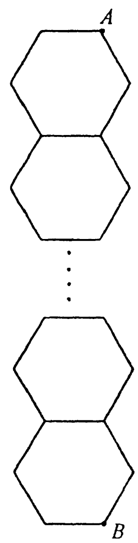
a) Let n be a positive integer. Explain why the maximum length of a repeating decimal of the form $\frac{1}{n}$ is $n-1$. (3 points)

b) Find all prime numbers p such that their reciprocals have repeating decimals of length 3 or less. Prove that your list is exhaustive. (6 points)

c) Let p be any prime greater than 3. Prove that the sum of the digits in one block of the repeating digits of $\frac{1}{p}$ is divisible by 9. (6 points)

7. In problems (a) and (b), the problem is to determine the number of distinct paths from A to B along hexagonal grids subject to the following rules: 1) the path can never move upward and 2) the path can never retrace a segment.

a) This hexagonal grid consists of a column of n hexagons stacked one on top of another as shown. Determine the number of paths from A to B in terms of n. Explain and justify your answer. (6 points)



b) This hexagonal grid consists of 3 columns of $n-1$, n, and $n-1$ hexagons respectively as shown. Determine the number of paths from A to B in terms of n. Explain and justify your answer. (8 points)