FORTY-SECOND ANNUAL OLYMPIAD

HIGH SCHOOL PRIZE COMPETITION

IN MATHEMATICS

2005 – 2006

Conducted by

The Massachusetts Association
of
Mathematics Leagues
(MAML)

Sponsored by

The Actuaries' Club of Boston

FIRST LEVEL EXAMINATION

Tuesday, October 25, 2005

Olympiad Level 1 Examination 2005

1.) If 5 gallons of a $Y\%$ salt solution are mixed with 2 gallons of a $4.5(Y)\%$ salt solution, what percent of the final mixture is salt?

A) $\frac{1}{14}Y$ B) $\frac{1}{2}Y$ C) $\frac{3}{4}Y$ D) $\frac{11}{14}Y$ E) $2Y$

2.) Given $a=\frac{4}{b}$ and $b=5-\frac{2}{c}$, find c expressed in terms of a.

A) $\frac{2a}{5a-4}$ B) $\frac{2a}{5a+4}$ C) $\frac{2a}{4a-5}$ D) $\frac{2}{5a-4}$ E) $\frac{2}{5a+4}$

3.) The graph of $\{x: |3-4x| \leq 5 \text{ and } |8-9x| \geq 9\}$ on the number line consists of two line segments. Find the length of the longer of these two segments.

A) $\frac{2}{18}$ B) $\frac{3}{18}$ C) $\frac{5}{18}$ D) $\frac{7}{18}$ E) $\frac{8}{18}$

4.) In a particular 365 day calendar year, October 2nd fell on a Tuesday. In the next calendar year (also 365 days), November 2nd will fall on which day of the week?

A) Sunday B) Tuesday C) Wednesday D) Friday E) Saturday

5.) A car travels a certain distance at 50 mph. It travels this same distance in 10 minutes less time than at 30 mph. What is the number of miles in this distance?

A) $12\frac{1}{2}$ B) $12\frac{2}{3}$ C) $12\frac{3}{4}$ D) 15 E) 18

6.) A right circular cone has an altitude of 12 inches and a diameter of 10 inches. The cone is cut by a plane parallel to its base and 8 inches from its vertex, forming a smaller cone. Determine the number of square inches in the lateral surface area of the smaller cone.

A) $\frac{130\pi}{3}$ B) $\frac{260\pi}{9}$ C) $\frac{8\sqrt{61}\pi}{9}$ D) $\frac{260\pi}{3}$ E) $\frac{260\pi}{27}$

7.) Find the area bounded by the graphs of $y - \frac{3}{2} = -|x+3|$, $y = 0$, and $y = -3$.

A) $\frac{9}{4}$ B) 9 C) 12 D) 18 E) 36

8.) If $\frac{3x-7y}{4x+5y} = 8$, find the value of $\frac{y-x}{2y}$.

A) $-\frac{29}{38}$ B) $-\frac{4}{9}$ C) $-\frac{38}{29}$ D) $\frac{29}{38}$ E) $\frac{38}{29}$

9.) A right prism is inscribed in a sphere of radius 10 inches. The bases of the prism are equilateral triangles and its height is 16 inches. Find the number of cubic inches in the volume of the prism.

A) $216\sqrt{3}$ B) $432\sqrt{3}$ C) $864\sqrt{3}$ D) 864 E) 1728

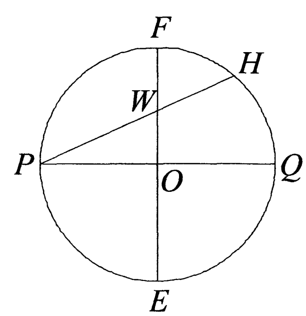
10.) Sean, Gautham, Andrew, and Nick go to the movies, and whenever anyone purchases an item, the cost is split evenly among the four at a later time. While they are there, Sean buys the tickets for $40, Gautham buys the candy for $10, Andrew buys the popcorn for $15, and Nick buys the soda for $15. Amidst the confusion of who is owed what amount of money, Nick and Gautham both give Sean $10. How much money does Andrew owe, and to whom?

A) $10 to Sean B) $10 to Nick C) $5 to Gautham D) $5 to Nick E) $10 to Gautham

11.) The Olympiad Level 1 Exam has 30 questions. A correct answer is worth 5 points, an answer left blank is worth 1 point, and an incorrect answer is worth 0 points. How many different scores are possible?

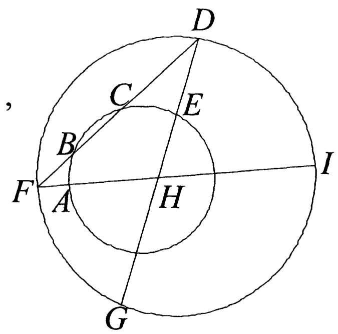
A) 141 B) 142 C) 143 D) 144 E) 145

12.) In circle O of radius 5 (as shown), $\overline{PQ} \perp \overline{EF}$ and $PH = 8$. Find the length of $\overline{EW}$.



A) $6\frac{1}{4}$ B) $6\frac{1}{2}$ C) $6\frac{3}{4}$ D) $8\frac{1}{4}$ E) $8\frac{3}{4}$

13.) Given two circles with one in the interior of the other (as shown). $\overline{FI}$ contains the center of the smaller circle, the measure of arc FG is 62°, $m\angle FHG = 70^\circ$, and the measure of arc AB is 36°.
Find the number of degrees in the measure of arc BC.



A) 25 B) 30 C) 35 D) 40 E) 45

14.) Let S_n be the set of three- digit numbers in base n. For how many values of n is there an element in S_n which has the same value as the base 10 number 2005?

A) 32 B) 33 C) 34 D) 35 E) 36

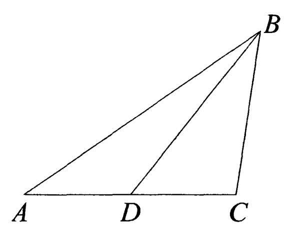
15.) Let a_n be the number of distinct positive factors of 2005^n (including 1 and 2005^n).
Find $\sum_{k=0}^{13} a_k$.

A) 105 B) 818 C) 819 D) 1014 E) 1015

16.) Abby, Brenda, and Carla are students in a school. Abby has 30 friends, Brenda has 20 friends, and Carla has 45 friends. There are less than 3 students who are friends with all 3 of the girls. Friendship is mutual. Find the minimum possible number of students in the school.

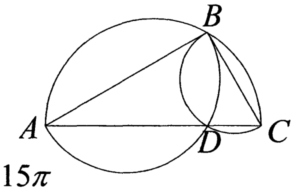
A) 45 B) 46 C) 47 D) 48 E) 49

17.) In ΔABC (as shown), $AC = 4$, D is the midpoint of $\overline{AC}$ and $m\angle CBD = 30^\circ$. Find the maximum length of $\overline{AB}$.



A) $2\sqrt{6}$ B) $3\sqrt{3}$ C) $3\sqrt{2}+1$ D) $2\sqrt{3}+2$ E) $\frac{5}{3}\sqrt{3}+3$

18.) Given ΔABC (as shown) with $AC = 12$ and $BC = 6$. Arcs ABC, ADB, and BDC are all semicircles. Find the area of the region common to all three semicircles.



A) $6\pi - 9\sqrt{3}$ B) $\frac{15\pi}{2} - 9\sqrt{3}$ C) $6\pi - \frac{9\sqrt{3}}{2}$ D) $15\pi - 18\sqrt{3}$ E) $\frac{15\pi}{2}$

19.) Find the remainder when $\sum_{n=1}^{2005} n!$ is divided by 2002.

A) 3 B) 782 C) 1007 D) 1078 E) 1881

20.) How many positive integers less than 2005 satisfy none of the following 3 conditions:
(i) Divisibility by 2.
(ii) Divisibility by 7.
(iii) Being a perfect square.

A) 815 B) 840 C) 842 D) 983 E) 985

21.) Bill chooses two cards at random from a standard deck of 52 cards. They are both spades. He then chooses 5 more cards at random. The probability that at least 5 of his 7 cards are spades is $\frac{N}{{}_{50}C_5}$.
Determine the value of N.

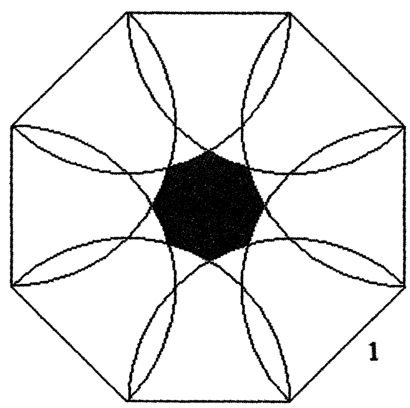
A) 122265 B) 129940 C) 135597 D) 141118 E) 142250

22.) Given a and b are positive numbers unequal to 1, $\log_b(a^{\log_2 b}) = \log_a(b^{\log_4 a})$, and $\log_a(p-(b-9)^2) = 2$. Find the minimum value for p.

A) $\frac{35}{4}$ B) $\frac{37}{4}$ C) $\frac{39}{4}$ D) $\frac{41}{4}$ E) $\frac{43}{4}$

23.) 8 partial circles of radius one are drawn (as shown), each centered on one of the vertices of a regular octagon with side length one. The area of the shaded region can be written in the form $\frac{a+b\sqrt{2}-c\sqrt{3}-\pi}{d}$, where a, b, c, and d are whole numbers. Find the value of $a+b+c+d$.

A) 18 B) 21 C) 24 D) 27 E) 30



24.) Given $0 < x < \frac{\pi}{2}$, $0 < y < \frac{\pi}{2}$, and $\frac{\cos x + \sin y}{\cos y - \sin x} = \cot x$, then $y = a\pi - bx$.
Find the value of $2a+b$.

A) 2 B) 3 C) 4 D) 5 E) 6

25.) Given $f(z) =$ the real part of a complex number z. For example, $f(3-4i)=3$.

If a is a positive integer, find $\sum_{n=1}^{6a} \log_2 \left| f\left(\left(1+i\sqrt{3}\right)^n\right) \right|$.

A) a^2+16a B) $36a-19$ C) $54a-37$ D) $18a^2-a$ E) $36a^2-19a$

26.) Find the sum of all roots to the equation $0=\sin\left(\pi \log_3\left(\frac{1}{x}\right)\right)$ where $0<x<2\pi$.

A) 3 B) 4 C) $\frac{13}{3}$ D) $\frac{40}{9}$ E) $\frac{9}{2}$

27.) The real roots of the equation $2x^3-19x^2+57x+k=0$ form an increasing geometric sequence x_1, x_2, x_3. Find the value of x_3.

A) 3 B $\frac{9}{2}$ C) $\frac{15}{3}$ D) $\frac{18}{5}$ E) 12

28.) A line is drawn through the point (5,0) so that it is tangent to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ at a point below the x-axis. Find the value of the slope of this line.

A) $\frac{3}{5}$ B) $\frac{\sqrt{21}}{7}$ C) $\frac{\sqrt{78}}{13}$ D) $\frac{2}{3}$ E) $\frac{\sqrt{91}}{13}$

29.) Boys and/or girls sit along a row of chairs. A boy must always sit next to at least one other boy and a girl must always sit next to at least one other girl. The row begins with a boy. If there are 15 chairs, how many possible gender arrangements are there?
For example: *BBBGGGGBBGGGGBB* is a possible arrangement.

A) 120 B) 292 C) 377 D) 428 E) $\frac{14!}{2}$

30.) Let $a_k=\left(\frac{1}{2}\right)^k+\left(\frac{i}{2}\right)^k$, where k is a positive integer. These numbers are plotted on the complex plane at the corresponding points A_1, A_2, A_3, etc. Let A_kA_{k+1} represent the distance between the points A_k and A_{k+1}. The value of the infinite sum $A_1A_2+A_2A_3+A_3A_4+\ldots.$ can be written as $\frac{a\sqrt{b}+c+\sqrt{d}}{e}$, where a, b, c, d, and e are integers and the radicals are in simplest form. Determine the sum : $a+b+c+d+e$.

A) 66 B) 68 C) 70 D) 72 E) 74