FORTY-FIRST ANNUAL OLYMPIAD

HIGH SCHOOL PRIZE COMPETITION

IN MATHEMATICS

2004 – 2005

Conducted By

The Massachusetts Association
of
Mathematics Leagues
(MAML)

Sponsored By

The Actuaries' Club of Boston

FIRST LEVEL EXAMINATION

Tuesday, October 26, 2004

1. If $9^x + 9^x + 9^x = 27^y$, then which one of the following statements is true?

A) $y = x$ B) $y = 2x$ C) $y = 3x$ D) $y = \frac{2x+3}{3}$ E) $y = \frac{2x+1}{3}$

2. A two-digit prime number N has two properties: 1) its ten's digit is two more than its unit's digit and 2) if N's digits are reversed, the result is also prime. Find the sum of all numbers N satisfying these properties.

A) 102 B) 118 C) 124 D) 128 E) 181

3. In convex pentagon $JKLMN$, $m\angle J = m\angle K = m\angle L = 100°$, and $\angle M \cong \angle N$. The bisectors of $\angle K$ and $\angle M$ meet at point P. What is the degree measure of $\angle KPM$?

A) 90° B) 100° C) 120° D) 140° E) 150°

4. How many positive integers less than 300 have the property that they are divisible by both 5 and 7, but not by either 4 or 25?

A) 4 B) 5 C) 6 D) 7 E) 8

5. A red cube and a green cube both have the whole numbers from 1 to 6 written on their faces, one number per face. The two cubes are tossed. In how many ways can the number on the top face of the red cube be at least two more than the number on the top face of the green cube?

A) 6 B) 8 C) 10 D) 12 E) 14

6. Bob runs 9 miles in the same time that Alice runs 15 miles. Bob's speed is 2.5 miles per hour less than Alice's speed. If Alice's speed is s miles per hour, which one of the following inequalities is true?

A) $4 < s < 5$ B) $5 < s < 6$ C) $6 < s < 7$ D) $7 < s < 8$ E) $8 < s < 9$

7. Katie has 5 times as many dollars as Joan. If Katie gives Joan \$20, Katie then has three times as many dollars as Joan. If d is the total amount of dollars that Katie and Joan have, which one of the following inequalities is true?

A) $100 < d < 150$ B) $150 < d < 200$ C) $200 < d < 250$ D) $250 < d < 300$ E) $300 < d < 350$

8. Two large inlet pipes and one small outlet pipe operating at the same time can fill a swimming pool in 16 hours. Each of the large pipes fills the pool four times as fast as the small pipe empties the pool. If the number of hours it would take one large pipe to fill the pool is t, which one of the following inequalities is true?

A) $10 < t < 15$ B) $15 < t < 20$ C) $20 < t < 25$ D) $25 < t < 30$ E) $30 < t < 35$

9. If one solution of $x^3 - 2x^2 + ax + 10 = 0$ is the additive inverse of another, then which one of the following inequalities is true?

A) $-50 < a < -40$ B) $-40 < a < -30$ C) $-30 < a < -20$ D) $-20 < a < -10$ E) $-10 < a < 0$

10. The diameter of a sphere equals the height of a right circular cylinder. If the base of the cylinder has a diameter equal to the radius of the sphere, what is the ratio of the volume of the cylinder to the volume of the sphere?

A) $\frac{1}{8}$ B) $\frac{1}{6}$ C) $\frac{1}{4}$ D) $\frac{3}{8}$ E) $\frac{3}{4}$

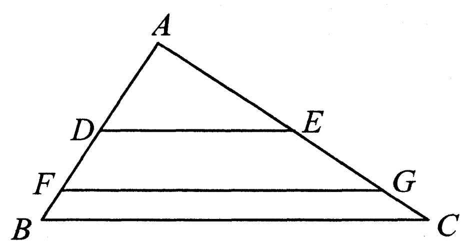
11. Given $\log_c 2 = w,\ \log_c 3 = y,$ and $\log_c 5 = z$, find $\log_{\frac{1}{c}} \sqrt{2.4}$ in terms of w, y and z.

A) $\frac{z-y-w}{2}$ B) $\frac{z-y-2w}{2}$ C) $\frac{2w+y-z}{2}$ D) $\frac{z+y-2w}{2}$ E) $\frac{w-y-z}{2}$

12. Point A is the vertex of the parabola whose equation is $y = x^2 - 2$. Points B and C are the intersections of the parabola with the circle whose equation is $x^2 + y^2 = 8$. Find the number of square units in the area of ΔABC.

A) 8 B) 9 C) 10 D) 12 E) 16

13. Given ΔABC, with $\overline{DE} \parallel \overline{FG} \parallel \overline{BC}$, $\frac{AD}{FB} = \frac{3}{1}$, and the ratio of the area of $DEGF$ to the area of ΔAGF $= \frac{16}{25}$. Find the ratio of the area of ΔADE to the area of $FGCB$.



A) $\frac{4}{3}$ B) $\frac{9}{8}$ C) $\frac{9}{10}$ D) $\frac{9}{11}$ E) $\frac{9}{16}$

14. The solutions to $9^y + 2^{x+1} = 3$ and $2^2 \cdot 3^y + 6 \cdot 2^x = 10$ are the ordered pairs of real numbers (a,b) and (c,d). Find the sum $b+d$.

A) -2 B) -1 C) 0 D) 1 E) 2

15. If 2004! is converted to base 11, then how many zeros will be at the end of the base 11 number?

A) 182 B) 196 C) 197 D) 199 E) 200

16. Twelve cards of which 5 are red, 4 are white, and 3 are blue are placed in a box. A red card is worth 1 point, a white card is worth 2 points, and a blue card is worth 3 points. Three cards are picked at random from the box. What is the probability that the total point value of the three cards is greater than or equal to 7? Find the result rounded to two decimal places.

A) 0.21 B) 0.23 C) 0.25 D) 0.27 E) 0.29

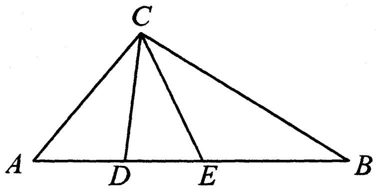
17. Given the equation $\sqrt{x+a}=\sqrt{x}+a$, $x\geq 0$, and $0<a\leq 1$, solve for x in terms of a.

A) $\frac{(1-a)^2}{4}$ B) $\frac{(1+a)^2}{4}$ C) $(1-a)^2$ D) $\frac{(1-a)^2}{16}$ E) $\frac{1+a^2}{4}$

18. The points of intersection of the lines $y=0$, $y=-2x+8$, and $y=mx$, $m>0$ are the vertices of a triangle. Find the area of the triangle in terms of m.

A) $\frac{16}{m+2}$ B) $\frac{32}{m+2}$ C) $\frac{8m}{m+2}$ D) $\frac{16m}{m+2}$ E) $\frac{32m}{m+2}$

19. In ΔABC, $\overline{CD}$ and $\overline{CE}$ trisect $\angle ACB$, as shown. Which one of the following statements is true?



A) $\frac{AD}{BE}=\frac{CA\cdot CD}{CE\cdot CB}$ B) $\frac{AE}{BD}=\frac{CA}{CB}$ C) $\frac{AD}{BE}=\frac{CD}{CE}$ D) $\frac{AD}{BE}=\frac{CA\cdot CE}{CD\cdot CB}$ E) $\frac{AE}{BE}=\frac{2CA}{CB}$

20. A parallelogram has consecutive sides of length 5 and 8 and a diagonal of length 9. Find the length of its other diagonal.

A) $\sqrt{145}$ B) 10 C) $\sqrt{97}$ D) $\sqrt{95}$ E) $\sqrt{91}$

21. In a geometric sequence of real numbers, the first term is 2 and the sum of the reciprocals of its third and fourth terms is 2 more than its second term. Find the sum of all possible values for its seventh term.

A) $\frac{1}{8}$ B) $\frac{3}{4}$ C) 2 D) $\frac{17}{8}$ E) 3

22. Given $i=\sqrt{-1}$ and $(x+yi)^2=45+ai$, where x and y are positive integers, then the smallest possible value for a satisfies which one of the following inequalities?

A) $10<a<20$ B) $20<a<30$ C) $30<a<40$ D) $40<a<50$ E) $50<a<60$

23. Given $f(x)=\dfrac{x^4-7x^2+9}{x-\frac{3}{x}+1}$. Its zeros are of the form $\frac{a\pm\sqrt{b}}{c}$, where a, b, and c are positive integers. Find the sum $a+b+c$.

A) 14 B) 15 C) 16 D) 17 E) 18

24. Four married couples sit at a round table for dinner. How many different arrangements are possible, if each couple occupies adjacent seats and Al, as one of the eight, picks the seat nearest the window?

A) 240 B) 180 C) 120 D) 96 E) 48

25. There are two possible values of A in the solution of the matrix equation $\begin{pmatrix} 2A+1 & -5 \\ -4 & A \end{pmatrix}^{-1} \cdot \begin{pmatrix} A-5 & B \\ 2A-2 & C \end{pmatrix} = \begin{pmatrix} 14 & D \\ E & F \end{pmatrix}$. Find the absolute value of the difference of these two solutions. A, B, C, D, E, and F are real numbers.

A) $\frac{8}{3}$ B) $\frac{11}{3}$ C) $\frac{13}{3}$ D) $\frac{17}{3}$ E) $\frac{19}{3}$

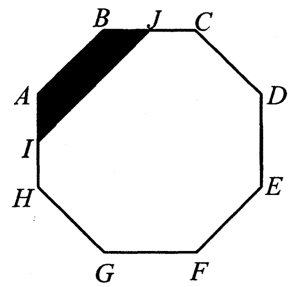
26. Consider the recursive sequence

$$a_n = \begin{cases} 2 & \text{if } n=1 \\ a_{n-1}+i & \text{if } n \text{ is even} \\ ia_{n-1} & \text{if } n \text{ is odd and } n>1 \end{cases}$$

where $i=\sqrt{-1}$. Evaluate the sum $\sum_{k=1}^{2004} a_k$.

A) $-2004+4i$ B) $-998+6i$ C) $-1000+2i$ D) $-1000+6i$ E) $-998+4i$

27. Given regular octagon $ABCDEFGH$, points I and J are the midpoints of sides $\overline{AH}$ and $\overline{BC}$, respectively. Find the ratio of the shaded area $ABJI$ to the area of the octagon.



A) $\frac{3-\sqrt{2}}{16}$ B) $\frac{1+\sqrt{2}}{16}$ C) $\frac{1}{16}$ D) $\frac{1}{18}$ E) $\frac{3+\sqrt{2}}{16}$

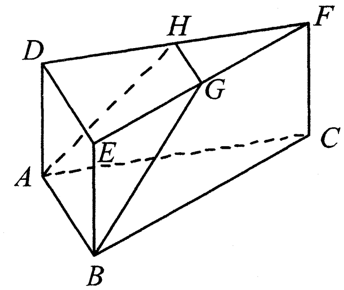
28. If 48! has n positive integer factors, how many positive integer factors does 49! have?

A) $3n$ B) $2n$ C) $\frac{3}{2}n$ D) $\frac{4}{3}n$ E) $\frac{9}{7}n$

29. There is a function $f(x)=ax^3+bx^2+cx+d$, where a, b, c, and d are integers and $a>0$, such that $f\left(\sin\left(\frac{\pi}{18}\right)\right)=0$. Find the smallest possible value for $f(1)$.

A) 1 B) 2 C) 3 D) 4 E) 5

30. Given right prism with a triangular base ABC with $AB=10$, $AC=BC=13$, and $CF=6$. Points H and G are the midpoints of $\overline{FD}$ and $\overline{FE}$ respectively. Find the volume of the solid with vertices. A, B, E, G, H, and D.



A) 180 B) 150 C) 135 D) 120 E) 100