FOURTIETH ANNUAL OLYMPIAD HIGH SCHOOL

PRIZE COMPETITION IN MATHEMATICS

2003 – 2004

Conducted By

The Massachusetts Association of Mathematics Leagues (MAML)

Sponsored By

The Actuaries' Club of Boston

FIRST LEVEL EXAMINATION

Tuesday, October 28, 2003

1. Simplify: $\dfrac{\dfrac{1}{64}+\dfrac{1}{128}+\dfrac{1}{256}}{\dfrac{1}{1024}+\dfrac{1}{2048}+\dfrac{1}{4096}} \cdot \dfrac{1+\dfrac{1}{2}+\dfrac{1}{4}}{\dfrac{1}{8}+\dfrac{1}{16}+\dfrac{1}{32}}$

(1) $\frac{1}{2}$ (B) $\frac{1}{8}$ (C) 4 (D) 2 (E) 8

2. The expression $\left(3^2\right)^3-4$ is equal to a perfect square times a prime number. What is the prime number?

(A) 25 (B) 19 (C) 13 (D) 29 (E) 7

3. This morning, five hundred dollars was deposited into a bank account that pays no interest. Every 3rd day after today, \$12 is withdrawn. Every 17th day after today, \$25 is withdrawn. Every 51st day after today, \$100 is deposited. How many days after today will the account have a negative balance?

(A) 131 (B) 132 (C) 133 (D) 134 (E) 135

4. In how many ways can two dollars be paid using only nickels dimes and quarters where the absolute difference between the number of quarters and the number of nickels is equal to two.

(A) 9 (B) 10 (C) 11 (D) 12 (E) 13

5. Amy works at a toy manufacturing company and her job is to make toy cars. After 5 hours, she made 4 toy cars. After 24 hours, she made 18 toy cars. Starting with the 5th hour, she makes the toy cars at a constant rate. During which hour will she make the 40th toy car?

(A) 40th (B) 35th (C) 52nd (D) 53rd (E) 50th

6. What is the smallest four digit number that is the product of prime numbers A, BC, and DE where A, B, C, D, and E are distinct digits?

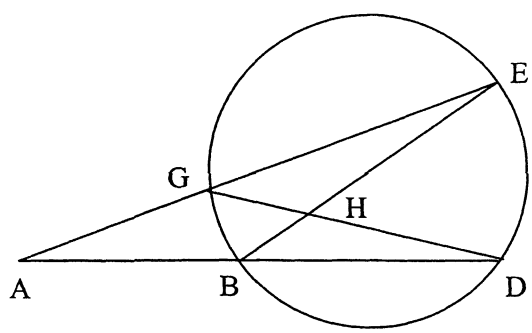
(A) 1234 (B) 1034 (C) 1222 (D) 1833 (E) 1066

7. Three red cubes have the same weight as six blue cubes, two yellow cubes have the same weight as five blue cubes, and four white cubes have the same weight as six blue cubes. How many blue cubes are needed to match the total weight of four red, two yellow, and two white cubes?

(A) 10 (B) 14 (C) 20 (D) 16 (E) 5

8. In the diagram at the right, $m\angle A = 60^\circ$ and $m\angle GHE = 100^\circ$. Find the $m\widehat{GB}$. (Note: the diagram is not drawn to scale.)

(A) 10 (B) 20 (C) 30
(E) 40 (F) 50



9. The difference between the sum of 25 consecutive integers and the sum of the 10 consecutive integers that precede them is 1900. Find the largest of these numbers.

(A) 93 (B) 95 (C) 126 (D) 129 (E) 127

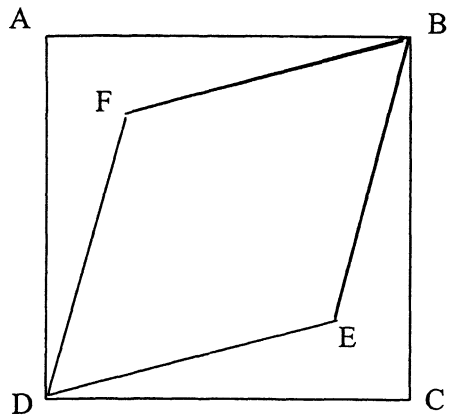
10. MN and PQ are distinct non prime numbers each of which is divisible by a two digit prime number. What is the largest possible product of MN and PQ?

(A) 9405 (B) 8712 (C) 9504 (D) 9012 (E) 7402

11. Consider the quadratic equation, $ax^2 + bx + c = 0$, where a, b, and, c are positive integers with $b^4 - 16a^2c^2 = 65$. The roots of this quadratic:

(A) are equal. (B) are irrational. (C) are imaginary.
(D) rational (E) cannot be determined.

12. Each side of square ABCD is 9 units long. The obtuse angle of rhombus BEDF measures 120°. What is the area of rhombus BEDF?



(A) 81 (B) $81\sqrt{3}$
(C) $18\sqrt{2}$ (D) $27\sqrt{3}$
(E) $27\sqrt{2}$

13. A block of wood is in the shape of a cube. Each edge of the cube measures 10 inches. Two holes in the shape of right cylinders of height 10 inches are drilled such that the radius of the smaller hole is half the radius of the larger hole. The holes are drilled so that there is no overlap. To the nearest half inch, how many inches should the radius of the larger hole be so that the volume of the cube with the holes is 65% of the volume of the original cube?

(A) 3 (B) 4 (C) 1.5 (D) 3.5 (E) 2.5

14. The five-digit base 10 number *babac* is a multiple of 25. How many distinct numbers are possible if the *a, b,* and, *c* can only take on the values 1, 2, 3, 4, 5, 6, or 7?

(A) 12 (B) 10 (C) 17 (D) 14 (E) 13

15. How many three-digit numbers with exactly two repeating digits are a multiple of nine?

(A) 36 (B) 18 (D) 12 (D) 20 (E) 21

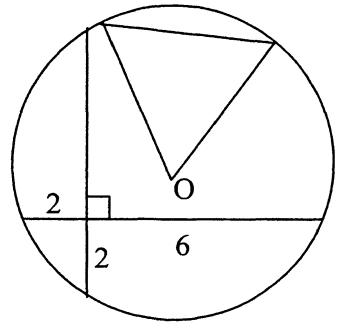
16. Two bins are filled with colored marbles. The first bin contains 3 blue and 7 black marbles. The second bin contains 5 blue and 9 white marbles. One marble is chosen at random from the first bin and put into the second bin. One marble is then chosen from the second bin. What is the probability that the marble chosen from the second bin is blue?

(A) $\frac{35}{53}$ (B) $\frac{3}{25}$ (C) $\frac{31}{81}$ (D) $\frac{53}{150}$ (E) $\frac{97}{150}$

17. Given the linear functions f and g, such that $f(x+1)=2+3x$, and, $g(x+2)=-5-3x$,
find $\dfrac{f(x)}{g(x)}$.

(A) $x+1$ (B) -1 (C) $\dfrac{1}{x+1}$ (D) $\dfrac{3x-1}{3x+1}$ (E) $\dfrac{1-3x}{x}$

18. The circle with center O in the diagram at the right has perpendicular chords with the lengths shown. Find the area of the equilateral triangle. Note: The diagram may not be drawn to scale.



(A) $2\sqrt{3}$ (B) $3\sqrt{3}$ (C) $\dfrac{\sqrt{3}}{2}$

(D) $6\sqrt{3}$ (E) $5\sqrt{3}$

19. If $\log_a 21 = x$ and $\log_a 49 = y$, which of the following is equivalent to $\log_a 27$ in terms of x and y?

(A) $x-\dfrac{y}{2}$ (B) $3x-\dfrac{y}{2}$ (C) $x-\dfrac{3y}{2}$ (D) $2x-\dfrac{y}{2}$ (E) $3x-\dfrac{3y}{2}$

20. Simplify $\dfrac{\sin^2\left(\frac{x}{2}\right)-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)-\cos^2\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)-1}$.

(A) $\cos\left(\dfrac{x}{2}\right)$ (B) $1+\sin x$ (C) $1+\tan x$ (D) $1+\sin\left(\dfrac{x}{2}\right)$ (E) $1-\tan x$

21. Each side of a regular hexagon is 4 cm log. The midpoints of the sides are joined consecutively to form another hexagon. This process is continued indefinitely. Determine the number of square centimeters in the total area of all such hexagons.

(A) $8\sqrt{3}$ (B) $24\sqrt{3}$ (C) $42\sqrt{3}$ (D) $72\sqrt{3}$ (E) $96\sqrt{3}$

22. A magic square is a $n \times n$ array filled with the integers $1,2,...,n^2$ with the property that each row and column of the array has the same sum. An unsuccessful attempt at placing the numbers produces a row with a sum of 12 and another row with a sum of 39. What must the value of n be?

(A) 5 (B) 6 (C) 4 (D) 7 (E) 10

23. Given the complex numbers $z_1 = 1 - ai$, $z_2 = 2 + ai$, and $z_3 = 1 + ai$ where a is a real number, such that $z_1 z_2 z_3$ is a real number, find $(z_1 z_2 z_3)^{10}$.

(A) 1024 (B) -1024 (C) 512 (D) 64 (D) 2048

24. On the surface of a right circular cylinder of radius $\frac{6}{\pi}$ and height 10; a spiral is drawn rising uniformly while making two complete turns. The length of the spiral is:

(A) $2\sqrt{61}$ (B) $\frac{360}{\pi}$ (C) 13 (D) 26 (D) 120

25. What is the sum of the values of x that satisfy this equation?
$(x^2 + 4x + 3)^{x^2 - 4x + 3} = (x^2 - 3x - 4)^{x^2 + 3x - 4}$

(A) -1 (B) -3 (C) 1 (D) -4 (E) 0

26. Given $x > 0$, $a > 0$, $x + a = \sqrt[6]{ax^2} + \sqrt[6]{a^5 x^4}$, and if x is expressed in terms of a, then all solutions are in the form a^n. Find the sum of all possible values for n.

(A) 3 (B) $\frac{13}{4}$ (C) $\frac{17}{6}$ (D) $\frac{11}{4}$ (E) $\frac{5}{2}$

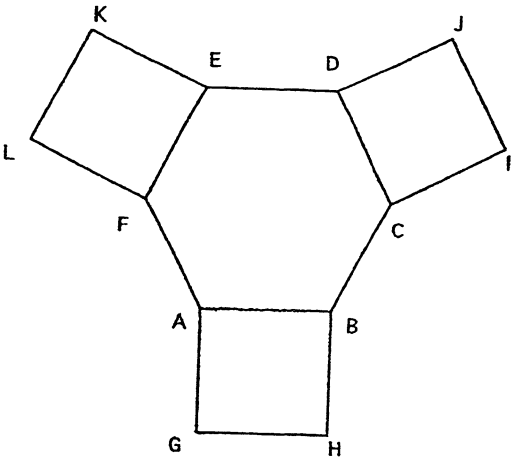
27. Which of the following is the best approximation of the sum of all values of θ in the interval $0^\circ \le \theta < 360^\circ$ that satisfy the equation $4 = 3\sec^2\theta(1 + \sin\theta)$?

(A) 450 (B) 600 (C) 173 (D) 200 (E) 135

28. Two hands of a clock are 4 inches and 5 inches long, respectively. At some time between 1:35 and 2:00, the tips of the hands are 8 inches apart. The time is closest to which of the following?

(A) 1:40 (B) 1:55 (C) 1:43 (D) 1:48 (E) 1:41

29. On regular hexagon ABCDEF where $\overline{AB}=6$, squares are drawn on $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ as shown. The length of $\overline{GK}$ in $\triangle GJK$ is:



(A) $3\sqrt{6}+9\sqrt{2}$ (B) $4\sqrt{6}+7\sqrt{2}$ (C) $12\sqrt{3}$
(D) $6+6\sqrt{3}$ (E) $12+6\sqrt{3}$

30. A $n\times n$ grid is filled by sequentially placing the natural numbers from 1 to n^2 in each cell from left to right. When the end of a row is reached, this process continues to the next row. This square has the property that the numbers in the cells on the diagonal containing the number 1 sum to 1379. What is the value of n?

(A) 15 (B) 16 (C) 21 (D) 14 (E) 13