# THIRTY-NINTH ANNUAL OLYMPIAD HIGH SCHOOL

# PRIZE COMPETITION IN MATHEMATICS

2002 - 2003

Conducted By

The Massachusetts Association
of
Mathematics Leagues
(MAML)

Sponsored By

The Actuaries' Club of Boston

## SECOND LEVEL EXAMINATION

Tuesday, April 1, 2003

<u>1. Find a Formula</u> (12 points total)

In each of the following problems, find and prove a general formula (covering infinitely many cases) of which the given equations are special cases:

(3 points) a. $2^2 + 3^2 + 2^2 3^3 = 7^2$

$5^2 + 6^2 + 5^2 6^3 = 31^2$

(4 points) b. $\frac{1}{1} + \frac{1}{3} = \frac{4}{3}$; $4^2 + 3^2 = 5^2$

$\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$; $8^2 + 15^2 = 17^2$

$\frac{1}{5} + \frac{1}{7} = \frac{12}{35}$; $12^2 + 35^2 = 37^2$

(5 points) c. $1^2 - 0^2 = 1^3$

$3^2 - 1^2 = 2^3$

$6^2 - 3^2 = 3^3$

$10^2 - 6^2 = 4^3$

<u>2. The Lottery</u> (11 points total)

In a minor league lottery a computer randomly chooses three distinct integers in the range of 1 to 40 inclusive. A player chooses six integers in this range, not knowing the computer's selections, and wins \$5000 if these six include all three of the computer's numbers. The player wins \$250 if her six include exactly two of the computer's three numbers.

(4 points) a. What is the probability (as a fraction in reduced form) of the player winning \$5000?

(4 points) b. What is the probability (as a fraction in reduced form) of winning \$250?

(3 points) c. What should a lottery ticket cost (to the nearest penny) to allow players to break even on average?

3. Its Divisible (13 points total)

(3 points) a. Show (without any dependence on a calculator) that the integer

$$5^{2003} + 6^{2003}$$

is divisible by 11.

(4 points) b. Show that for every positive integer $n$, the integer

$$(n^3 - n)(5^{8n+4} + 3^{4n+2})$$

is divisible by 3804.

(6 points) c. A shop distributes 9999 tickets, numbered from 0001 to 9999, among its customers. If the sum of the first two digits is equal to the sum of the last two, the ticket is a winner. For example, the ticket with number 0743 is a winner because $0 + 7 = 4 + 3$. Prove that the sum of the numbers on all winning tickets is divisible by 101.

4. <u>Log Jam!</u> (13 points total)

(4 points) a. Solve the equation

$$\sqrt{x^{\log_r \sqrt{x}}} = r$$

for $x$ in terms of $r$, given that $r > 1$.

(4 points) b. Given that $x$ and y are positive numbers and $r > 1$, solve the equation

$$\log_r x + \log_r y = \log_{r^2}(x^2 + y^2)$$

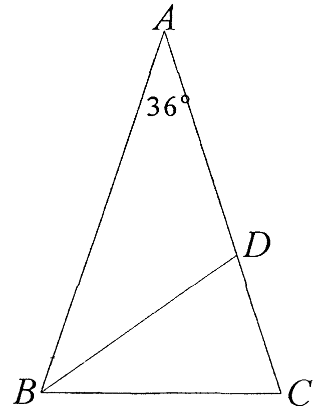
for $y$ as a function of $x$ and $r$, expressed as simply as you can. What is the domain of the function?

(5 points) c. If

$$\frac{(xy)^{\log_y (xy)}}{x^2 y} = x^p,$$

express $p$ as simply as you can as a function in terms of $x$ and $y$.

5. Pent-Up Knowledge (14 points total)

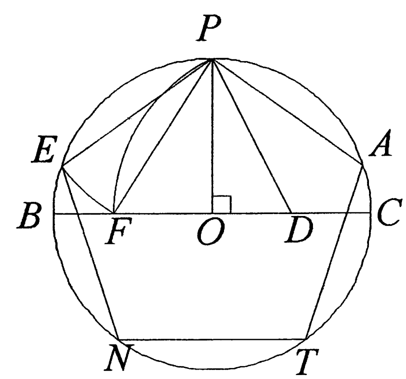



(5 points) a. Given isosceles $\Delta ABC$ with vertex angle $A$ measuring 36° and $\overline{BD}$ bisecting $\angle ABC$. Prove that $\frac{BC}{AB}=\frac{\sqrt{5}-1}{2}$.

(2 points) b. Using the result from part (a), find the exact value of $\cos(72°)$.

(3 points) c. A regular pentagon is inscribed in a circle of radius *r*. Using the result of (b), the exact length of one side of the pentagon $= r\sqrt{\frac{a-\sqrt{b}}{c}}$, where *a*, *b*, and *c* are relatively prime positive integers. Find the triple $(a,b,c)$.

(4 points) d. Circle *O* has diameter $\overline{BC}$, $\overline{PO}\perp\overline{BC}$, $\overline{OD}\cong\overline{DC}$, $\overset{\frown}{PF}$ centered at *D*, and $\overset{\frown}{EF}$ centered at *P*. $\overline{PE}\cong\overline{EN}\cong\overline{NT}\cong\overline{TA}$. Prove that *PENTA* is a regular pentagon.



6. Sequences of Composites (12 points total)

(3 point) a. A sequence of integers begins

$$2, 21, 219, 2192, 21921, 219219, \ldots,$$

each term being formed by appending one more digit on the right side of the last term. Digit 2 is always followed by a 1, 1 by 9, and 9 by 2. The first term of the sequence is prime. Prove that every term after the first term is composite.

(4 points) b. A sequence of integers begins

$$10001, 100010001, 1000100010001, \ldots,$$

and each term is obtained by appending 0001 on the right of the last term. Observe that $10001 = (73)(137)$. Prove that every term in the sequence is composite.

(5 points) c. The first two terms ($n$ = 1, 2) of the sequence $3^n - n^2$ are prime. Show that all subsequent terms are composite.

7. <u>Algebraic Numbers</u> (13 points total)

A real number $r$ is called algebraic if there is a polynomial $P(x) = a_0 + a_1 x + \ldots + a_n x^n$, with integer coefficients $a_k$, such that $P(r) = 0$. In each of the following cases show that $r$ is algebraic by finding an explicit polynomial $P(x)$, with $a_n = 1$ and $n$ as small as possible, such that $P(r) = 0$.

(2 points) a. $r = 2\sqrt[3]{3}$

(3 points) b. $r = \sqrt[4]{2} + 1$

(3 points) c. $r = \sqrt{3} + \sqrt{5}$

(5 points) d. $r = 1 + \sqrt[3]{2} - \sqrt[3]{4}$

8. Trigonometric Inequality (12 points total)

(6 points) a. Prove that if $0 < x < \pi$ then $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x > 0$.

(6 points) b. A right triangle has legs $a$ and $b$ and hypotenuse $c$. If the acute angles of the triangle are $\alpha$ and $\beta$, prove that

$$\cos^2\left(\frac{\alpha-\beta}{2}\right) \geq \frac{2ab}{c^2}$$