Detailed Solutions to 2002 MAML Olympiad Level 1

1. (D)

$\left(2^4\right)^2 - 1 = \left(2^4 - 1\right)\left(2^4 + 1\right) = 15 \cdot 17 \Rightarrow$ largest prime factor of the number is 17.

2. (A)

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}} = 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{2}{5}}} = 1+\cfrac{1}{2+\cfrac{5}{12}} = 1+\frac{12}{29} = \frac{41}{29}.$$

3. (E)

$86 \div 3 = 28R2, 86 \div 5 = 17R1 \Rightarrow$ 28 times \$15 will be added and 17 times \$17 will be subtracted $\Rightarrow$ amount of money after 86 years = $10 + 28 \times 15 - 17 \times 17 = 141$.

4. (C)

$\frac{37+82+48+94+30+120+6A}{9} = 67 \Rightarrow 6A + 411 = 603 \Rightarrow A = 32 \Rightarrow 2A = 64 \Rightarrow 3A = 96$.
S = $\{30, 32, 37, 48, 64, 82, 94, 96, 120\} \Rightarrow$ median = 64.

5. (B)

Let d = number of dimes and q = number of quarters $\Rightarrow 10d + 25q = 500 \Rightarrow 2d + 5q = 100 \Rightarrow$
$d = \frac{100 - 5q}{2} \Rightarrow q = 0, 2, 4, \ldots, 20 \Rightarrow$ there are 11 combinations.

6. (D)

Based on the conditions, the number of inches of snow after each hour form an arithmetic sequence in which $a_1 = 1$ and $a_{48} = 36$. $36 = 1 + 47d \Rightarrow d = \frac{35}{47} \Rightarrow a_{36} = 1 + 35\left(\frac{35}{47}\right) \approx 27.1$ inches of snow.

7. (A)

If d_0 = units digit, d_1 = tens digit, d_2 = hundreds digit, and d_3 = thousands digit $\Rightarrow$
$1000d_3 + 100d_2 + 10d_1 + d_0 - \left(d_3 + d_2 + d_1 + d_0\right) = 999d_3 + 99d_2 + 9d_1 = 9\left(111d_3 + 11d_2 + d_1\right) \Rightarrow$
the new number is divisible by 9. Since the sum of the digits of any number divisible by 9 equals a number divisible by 9 and three of the digits add to 8, then the missing digit = 1.

8. (D)

The discriminant of the quadratic is less than 0 $\Rightarrow$ $a^2 - 24 < 0 \Rightarrow a = 0, \pm 1, \pm 2, \pm 3, \pm 4$
$\Rightarrow$ there are 9 possibilities for a.

9. (C)

Since $346,xy2$ is divisible by 4 $\Rightarrow$ $y = 1,3,5,7,9$. Since $346,xy2$ is also divisible by 9 $\Rightarrow$ $15 + x + y$ is divisible by 9 $\Rightarrow$ if $y = 1 \Rightarrow x = 2$, if $y = 3 \Rightarrow x = 0$ or 9, if $y = 5 \Rightarrow x = 7$, if $y = 7 \Rightarrow x = 5$, and if $y = 9 \Rightarrow x = 3 \Rightarrow$ there are 6 ordered pairs.

10. (A)

Let x = width of the rectangle $\Rightarrow 2x$ = its length $\Rightarrow$ $2x^2 = 64 \Rightarrow x^2 = 32 \Rightarrow$ $x = 4\sqrt{2}$; since the width of the rectangle = diagonal of the square $\Rightarrow$ side of the square = 4 $\Rightarrow$ its area = 16.

11. (A)

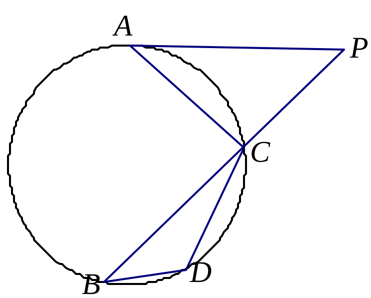
The sum of the first 100 positive integers = $\frac{100}{2}(1+100) = 5050$; the sum of one row or one column = $5050 \div 10 = 505$.

12. (C)

Consider the set of numbers from 1 to 30. The elements of this set not divisible by 2, 3, or 5 are 1, 7, 11, 13, 17, 19, 23, and 29, which are 8 numbers $\Rightarrow$ for every subsequent set of 30 whole numbers 8 are not divisible by 2, 3, or 5. $2002 \div 8 = 250$, remainder 2. Since the second number not divisible by 2, 3, or 5 is 7, then the 2002$^{\text{nd}}$ number of this type = $250 \times 30 + 7 = 7507$.

13. (B)

Since $m\angle BDC = 125° \Rightarrow m\widehat{BAC} = 250°$; $m\angle CAP = 40° \Rightarrow$
$m\widehat{AC} = 80°$; $m\widehat{AB} = 250° - 80° = 170°$;
$m\angle P = \frac{1}{2}\left(m\widehat{AB} - m\widehat{AC}\right) = \frac{1}{2}(170 - 80) = 45°$;
$m\angle ACP + m\angle P + m\angle CAP = 180° \Rightarrow m\angle ACP = 95°$.



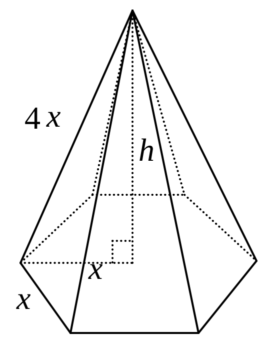
14. (B)

The number of zeros at the end of 150! = number of factors of 10 in 150! = number of factors of 5 in 150! . Counting the number of factors of 5: 5×1, 5×2, 5×3, …,5×30 = 30 factors of 5; 25×1, 25×2, 25×3,… ,25×6 gives you 6 more factors of 5. 125×1 gives you 1 more factor of 5 ⇒ there are 30 + 6 + 1 = 37 zeros at the end of the decimal expansion of 150!. Using the "greatest integer less than or equal to" function then a simple way to do the problem: $\left[\frac{150}{5}\right]+\left[\frac{150}{25}\right]+\left[\frac{150}{125}\right]=30+6+1=37$.

15. (A)

$f(2x+4)=6x+13=3(2x+4)+1 \Rightarrow f(x)=3x+1 \Rightarrow f(5-3x)+f(5x-1)=$
$3(5-3x)+1+3(5x-1)+1=6x+14$.

16. (B)



To find the height of the pyramid ⇒ $x^2+h^2=(4x)^2 \Rightarrow$
$h=x\sqrt{15}$. Since the volume of the pyramid = 60 ⇒
$\frac{1}{3}\cdot\frac{3x^2\sqrt{3}}{2}\cdot x\sqrt{15}=60 \Rightarrow \frac{3x^3\sqrt{5}}{2}=60 \Rightarrow x^3=8\sqrt{5} \Rightarrow x=2\sqrt[6]{5}$.

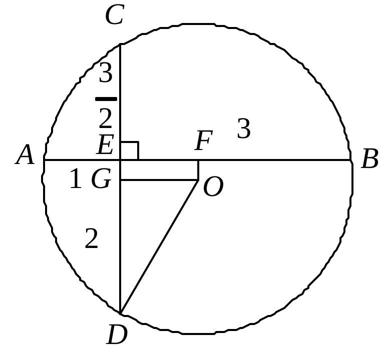
17. (C)

The probability A will be champion = $\frac{4}{7}$ and the probability B will be champion = $\frac{1}{5}$ ⇒ the probability that either one will be champion = $\frac{4}{7}+\frac{1}{5}=\frac{27}{35}$ ⇒ the odds of this happening = 27 to 8.

18. (B)

$AB_{13}=BA_{15} \Rightarrow 13A+B=15B+A \Rightarrow 12A=14B \Rightarrow 6A=7B$ and since A is a digit in base 13 or 15, $A<13$ ⇒ the largest and only value for A = 7.

19. (E)



Label the perpendicular chords $\overline{AB}$ and $\overline{CD}$ intersecting at E. Let O be the center of the circle. Draw perpendiculars $\overline{OF}$ and $\overline{OG}$ to each chord forming rectangle $GEFO$. By a theorem in geometry $AE\cdot EB=CE\cdot ED \Rightarrow 2CE=3 \Rightarrow CE=\frac{3}{2}$; F and G are midpoints of their respective chords ⇒ $AF=2$ and $GD=\frac{7}{4}$ ⇒
$EF=GO=1 \Rightarrow$ radius = $OD=\sqrt{1^2+\left(\frac{7}{4}\right)^2}=\sqrt{\frac{65}{16}}=\frac{\sqrt{65}}{4}$.

20. (E)

$$\left[\sin\left(\frac{5p}{2}+x\right)\right]\left[\cos\left(\frac{7p}{2}-x\right)\right]=\left(\cos\frac{4p}{3}\right)\left(\sin\frac{7p}{4}\right)\Rightarrow\left[\sin\left(\frac{p}{2}+x\right)\right]\left[\cos\left(\frac{3p}{2}-x\right)\right]=\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)\Rightarrow$$

using reduction or sum and difference formulas, $\cos x\left(-\sin x\right)=\frac{\sqrt{2}}{4}\Rightarrow 2\sin x\cos x=-\frac{\sqrt{2}}{2}\Rightarrow$

$\sin\left(2x\right)=-\frac{\sqrt{2}}{2}\Rightarrow 2x=\frac{5p}{4},\frac{7p}{4},\frac{13p}{4},\frac{15p}{4}\Rightarrow x=\frac{5p}{8},\frac{7p}{8},\frac{13p}{8},\frac{15p}{8}\Rightarrow$

sum of all solutions for $x = 5p$.

21. (D)

Since the circle passes through $\left(0,0\right)$ and $\left(2,0\right)$, its center must be on the line $x=1$. The center is also on the perpendicular bisector of $\left(0,0\right)$ and $\left(1,1\right)$ $\Rightarrow$ slope of this line is –1 and it passes through $\left(0.5,0.5\right)$ $\Rightarrow$ the equation of this second line is $y=-x+1$ $\Rightarrow$ center is $\left(1,0\right)\Rightarrow$ diameter is 2.
Note: Sketching the three points makes it obvious that the center is $\left(1,0\right)$.

22. (D)

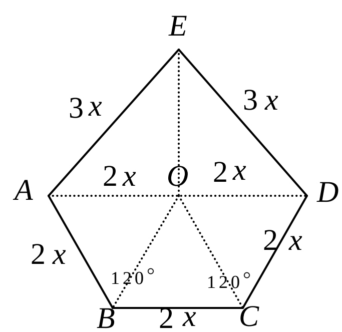
The sample space has ${}_{15}C_4$ elements. The successful events are 2 reds, 1 blue, 1 white, or 2 blues, 1 red, 1 white, or 2 whites, 1 red, 1 blue. Therefore the probability of having at least one of each color = $\frac{{}_6C_2\cdot{}_5C_1\cdot{}_4C_1+{}_5C_2\cdot{}_6C_1\cdot{}_4C_1+{}_4C_2\cdot{}_6C_1\cdot{}_5C_1}{{}_{15}C_4}=\frac{48}{91}$.

23. (A)

Draw $\overline{AD}$ and its midpoint O and $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$.
Now the pentagon is divided into three equilateral triangles and two right triangles $\Rightarrow$

$$3\cdot\frac{\left(2x\right)^2\sqrt{3}}{4}+2\cdot\frac{\left(2x\right)\left(x\sqrt{5}\right)}{2}=20\Rightarrow\ 3x^2\sqrt{3}+2x^2\sqrt{5}=20$$

$$\Rightarrow\ x=\sqrt{\frac{20}{3\sqrt{3}+2\sqrt{5}}}\Rightarrow\text{Perimeter }=12\sqrt{\frac{20}{3\sqrt{3}+2\sqrt{5}}}\approx 17.3\,.$$



24. (C)

$\log_2(x-1)+\log_2(x+2)-\log_2(3x-1)<1\Rightarrow$ domain for x from the intersection of the domains of all three logarithm terms is $x>1$. $\log_2(x-1)+\log_2(x+2)<\log_2(3x-1)+\log_2\left(2\right)\Rightarrow$

$\log_2\left((x-1)(x+2)\right)<\log_2\left(2(3x-1)\right)\Rightarrow x^2+x-2<6x-2\Rightarrow x^2-5x<0\Rightarrow x\left(x-5\right)<0$

$\Rightarrow\ 0<x<5$. Intersecting this inequality with $x>1\Rightarrow\ 1<x<5$.

25. (B)

$(x^2+3x+1)^{x^2+2x-8}=1$ is true if (i) the exponent = 0, (ii) the base = 1, or (iii) the base = –1 and the exponent is even.

(i) $x^2+2x-8=0 \Rightarrow (x+4)(x-2)=0 \Rightarrow x=-4,2$.

(ii) $x^2+3x+1=1 \Rightarrow x^2+3x=0 \Rightarrow x(x+3)=0 \Rightarrow x=-3,0$.

(iii) $x^2+3x+1=-1 \Rightarrow x^2+3x+2=0 \Rightarrow (x+1)(x+2)=0 \Rightarrow x=-2,-1$. Now check the exponent.

If $x=-2 \Rightarrow x^2+2x-8$ is even; if $x=-1 \Rightarrow x^2+2x-8$ is odd; therefore case (iii) produces the solution –2 only. The sum of all the solutions = $-4+2+(-3)+0+(-2)=-7$.

26. (E)

Let a = first term and d = constant difference between terms $\Rightarrow$

$\dfrac{a+3d}{a+6d}=\dfrac{2}{3} \Rightarrow 3a+9d=2a+12d \Rightarrow a=3d \Rightarrow \dfrac{100}{2}(2a+99d)=7000 \Rightarrow$

$2a+99d=140 \Rightarrow 105d=140 \Rightarrow d=\dfrac{4}{3} \Rightarrow a=4 \Rightarrow$ 2002nd term = $4+2001\left(\dfrac{4}{3}\right)=2672$.

27. (E)

$\sin\left(x+\dfrac{p}{4}\right)-\sin\left(x-\dfrac{p}{4}\right)=t \Rightarrow \left(\sin x\cos\left(\dfrac{p}{4}\right)+\cos x\sin\left(\dfrac{p}{4}\right)\right)-\left(\sin x\cos\left(\dfrac{p}{4}\right)-\cos x\sin\left(\dfrac{p}{4}\right)\right)=t \Rightarrow$

$2\cos x\left(\dfrac{\sqrt{2}}{2}\right)=t \Rightarrow \cos x=\dfrac{t}{\sqrt{2}} \Rightarrow x=\text{Arccos}\left(\dfrac{t}{\sqrt{2}}\right)$.

28. (B)

Let x and y be the lengths of the consecutive sides of the parallelograms $\Rightarrow$ $p=2x+2y \Rightarrow x+y=\dfrac{p}{2}$.

To find the length of the longer diagonal, use the Law of Cosines $\Rightarrow$

$d^2=x^2+y^2-2xy\cos120° \Rightarrow d^2=x^2+y^2+xy$. The area of the parallelogram = $xy\sin60°=\dfrac{xy\sqrt{3}}{2}$.

Since $x^2+2xy+y^2=\dfrac{p^2}{4}$, subtracting equations gives the result, $xy=\dfrac{p^2}{4}-d^2 \Rightarrow$

the area of the parallelogram = $\dfrac{\sqrt{3}}{2}\left(\dfrac{p^2}{4}-d^2\right)=\dfrac{\sqrt{3}}{8}\left(p^2-4d^2\right)$.

29. (B)

Let the coordinates of point $Q = (a, 2a-4) \Rightarrow$ the coordinates of point $R = (2a, 4a-4)$.
Using the determinant method to find areas of triangles,
[Note: The || || mean absolute value of the determinant.]

$$\frac{1}{2}\left|\begin{vmatrix} 4 & 0 & 1 \\ a & 2a-4 & 1 \\ 2a & 4a-4 & 1 \end{vmatrix}\right| = 6 \Rightarrow \left|4(-2a) + 4a^2 - 4a - (4a^2 - 8a)\right| = 12 \Rightarrow |4a| = 12 \Rightarrow a = \pm 3 \Rightarrow$$

$$Q = (3,2) \;\; and \;\; Q' = (-3,-10) \Rightarrow QQ' = \sqrt{6^2 + 12^2} = 6\sqrt{5}\,.$$

30. (A)

$z^3 = (1+ai)^3 = 1 + 3ai - 3a^2 - a^3 i$; since z^3 is a real number $\Rightarrow 3a - a^3 = 0$ and since $a > 0 \Rightarrow a = \sqrt{3} \Rightarrow z^3 = 1 - 3a^2 = 1 - 9 = -8$;

$$1 + z + z^2 + \ldots + z^{11} = \frac{1 - z^{12}}{1 - z} = \frac{1 - (z^3)^4}{1 - z} = \frac{1 - (-8)^4}{1 - (1 + i\sqrt{3})} = \frac{-4095}{-i\sqrt{3}} = -1365i\sqrt{3}\,.$$