1. How Big is that Polygon?

In convex polygon $A_1A_2A_3...A_n$, $\angle A_n = \angle A_1 = 90°$ and all of the remaining angles are equal. Sides $A_1A_2, A_2A_3, A_3A_4, ..., A_{n-1}A_n$ are all of length 1.

Compute (i) the length of A_nA_1, **and**
(ii) the exact area of the polygon, when:

(1 point) a. $n = 4$

(3 points) b. $n = 5$

(5 points) c. $n = 6$

2. Graph the Greatest

For any real number t, $\lfloor t \rfloor$ is the largest integer that is less than or equal to t.

(4 points) a. On six separate axes, for $-3 < x < 3$, sketch the following, clearly indicating which boundary points are parts of the graph and which are not:

i. $y = \lfloor x \rfloor^2$

ii. $y = \lfloor x^2 \rfloor$

iii. $\lfloor y \rfloor = x^2$

iv. $\lfloor y \rfloor = \lfloor x \rfloor^2$

(2 points) b. Compute the area of the graph of $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25$.

(3 points) c. List all positive integers $k \in \{1, 2, 3, \ldots, 20\}$ for which the area of the graph of $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = k^2$ is 4, and explain the reasons for your inclusions and exclusions.

3. My Calculator Broke ☹

(5 points) a. My broken calculator skips displaying the zero, so the display will show 16 if the answer is any of the following:

16, 106, 160, 1006, 1060, 1600, …

For each of the displays 11, 12, 13,…,19, list the products of a one-digit prime times a two-digit number which will result in that display.

(4 points) b. A repairman fixed the display on my calculator, but when he put it back together, he put the "plus" key where the "times" key should be and he put the "times" key where the "plus" key should be. Consequently, when I enter $2 \cdot 3 + 4$ the calculator displays 14 since it computed $2 + 3 \cdot 4$. Specify, with proof, the set of all triples (a, b, c) of positive integers for which the entry $a \cdot b + c$ will give the correct answer on my calculator.

4. Equal Rights for All Digits!

(3 points) a. $q = 71^{43}$ is an 80-digit number in its base ten representation. Prove that at least one of the ten digits appears at least nine times.

(6 points) b. If m and n are positive integers and each of the base ten digits appears an equal number of times in the base ten representation of m^n, then, besides m and n being the right size so that the number of digits in m^n is divisible by 10, what other properties must m and n have? **Prove** that m and n must have the properties that you state.

5. Beautify with Balloons!

(1 point) a. Balloons are to be attached to the three corners of a large triangular billboard. There are several balloons available in each of these six colors: red, white, blue, orange, green and yellow. Balloons of different colors are to be assigned to corners sharing a common edge. Compute the number of patterns of balloons possible, *and explain your computations*.

(2 points) b. Suppose the sign is rectangular and there are several balloons available in each of six colors. Balloons are to be attached to the four corners of the billboard such that balloons of different colors are to be assigned to the corners sharing a common edge. Compute the number of patterns of balloons possible, *and explain your computations*.

(3 points) c. Suppose the sign is pentagonal, several balloons in each of the six colors are available, and the balloons of different colors are to be assigned to corners sharing a common edge. Compute the number of patterns of balloons possible, *and explain your computations*.

(3 points) d. The climbing rose died. To decorate the dilapidated rose arbor for the party, balloons are to be attached to the five junctions *a*, *b*, *c*, *d*, and *e* of the remaining slats (see the figure below). There are several balloons available in each of n colors. Balloons of different colors are to be attached to corners sharing a common slat. Compute the polynomial $f(n)$ which will give the number patterns of balloons possible, *and explain your derivation of* $f(n)$.

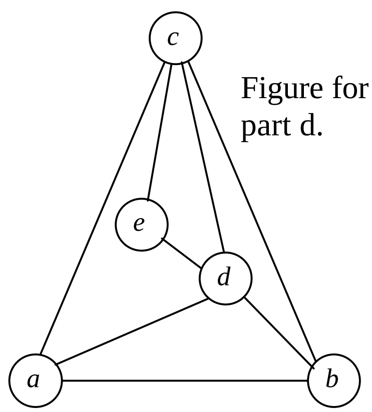

Figure for #5 part d.

6. What Pythagoras Wondered About

Squares *ABDE*, *BCFG*, and *ACHI* are constructed on the sides of triangle *ABC*, exterior to it. Sides *DE*, *FG* and *HI* are extended in both directions until they meet to form triangle *JKL*.

Compute the perimeter of triangle JKL in the following situations:

(1 point) a. If triangle *ABC* is equilateral with $AB = 1$.

(2 points) b. If triangle *ABC* is an isosceles right triangle with $AC = BC = 1$.

(2 points) c. If $AC = 3$, $BC = 4$, and $AB = 5$.

(4 points) d. If triangle *ABC* is isosceles with base $BC = a$ and sides $AB = AC = b$.

7. Eight is the Number

(2 points) a. Prove that if n is an odd integer, then 8 is a factor of $n^2 - 1$.

(2 points) b. In chess a Queen can attack any piece in its row or column or on either of the two $45°$ diagonals on the board. Give an arrangement of 8 Queens on an 8 by 8 chessboard such that no Queen is attacking any of the other Queens.

(2 points) c. How many different sums can one obtain by choosing 8 different numbers from the set $\{1, 2, 3, \ldots, 88\}$ and adding them together? *Justify your answer.*

(3 points) d. A palindrome is a sequence of symbols which reads the same forwards as backwards. For example, 2002 is a base-10 number that is a palindrome and LEVEL and PEEP are words in the English language that are palindromes. Prove that every integer that can be expressed as an 8-digit palindrome of non-zero digits in the base-7 number system is divisible by 8.

8. How Stupid is Her Hamster?

A biologist designs an experiment to approximate her hamster's ability to remember. In the experiment, the hamster is put into a cage with three identical exits which we will call *A*, *B*, and *C*. Exits *A* and *B* leads to tubes which are 3 and 4 meters long, respectively, and each tube ends in a steep slide that returns the hamster to the cage. Exit *C* leads to a tube 5 meters long which leads to food.

(1 point) a. A hamster with complete memory will first choose one of the three exits at random, each with the same probability. If it finds itself back in the cage after the first choice, it will choose one of the remaining exits, each with the same probability. If it finds itself back in the cage after two choices, it will choose the remaining exit and find food. What is the expected value (average) of the number of meters a hamster with complete memory will have to run before finding food? Show your computations.

(2 points) b. A hamster with absolutely no ability to remember will choose one of the three exits above at random, each with the same probability, every time it is in the cage. What is the expected value of the number of meters such a hamster will have to run before finding food? *Explain your computations.*

(3 points) c. In this problem we are back to a hamster with compete memory. Suppose that exits *A*, *B*, and *C* lead to tubes of length *a*, *b*, and *c*, respectively, with exits *A* and *B* leading to steep slides returning to the cage, and exit *C* leading to food. Suppose furthermore that the hamster shows preferences for exits *A*, *B* and *C* in the ratio *P:Q:R*. (In the situation where only *A* and *B* are available, for example, the preference will be in the ratio *P:Q*, etc.) Express the expected value of the number of meters such a hamster will have to run before finding food in terms of *a*, *b*, *c*, *P*, *Q*, *R*.

(3 points) d. Suppose that exits *A*, *B* and *C* lead to tubes of length *a*, *b*, and *c* respectively, with the first two exits leading to steep slides returning to the cage, and the third to food, as above. Suppose a hamster with absolutely no ability to remember for some reason will choose exits *A*, *B* and *C* at random with probabilities *p*, *q* and *r*, where $p, q, r > 0$ and $p + q + r = 1$. Express the expected value of the number of meters such a hamster will have to run before finding food in terms of the five variables a, b, c, p, q.