NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2003

Round 1 Arithmetic and Number Theory

1. ______________________

2. ______________________

3. ______________________

1. If a♦b = ab – 1 and a♠b = a + b^2, find 5♦(4♠3)

2. If $\dfrac{2^2 \cdot 3^3 \cdot 5^5 \cdot 7^7}{(20 \cdot 30 \cdot 50 \cdot 70)^3} = 2^a \cdot 3^b \cdot 5^c \cdot 7^d$ for integers *a*, *b*, c, and *d*, find the value of the sum $a + b + c + d$.

3. If *n* is a positive integer less than 10,000, for how many values of *n* is $\sqrt[3]{4n}$ an integer?

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2003

Round 2   Algebra 1

1. ______________________

2. ______________________

3. ______________________

1. Factor completely: c(a + b) – 4(a + 2b) + 2b(c – 2)

2. Measuring a certain distance by a meter stick that was 10% too short gave a length that was 40 cm too long. What was the actual distance in meters?

3. For $x > y > 0$ determine the greater value of $\frac{x}{y}$ if $\frac{x^3 - y^3}{x^3 - 3x^2y + 3xy^2 - y^3} = 4$.

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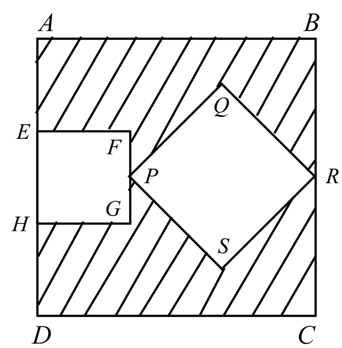
Round 3 – Geometry

1. ________________________

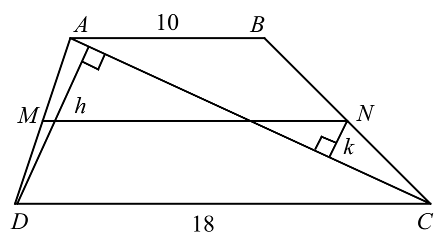
2. ________________________

3. ________________________

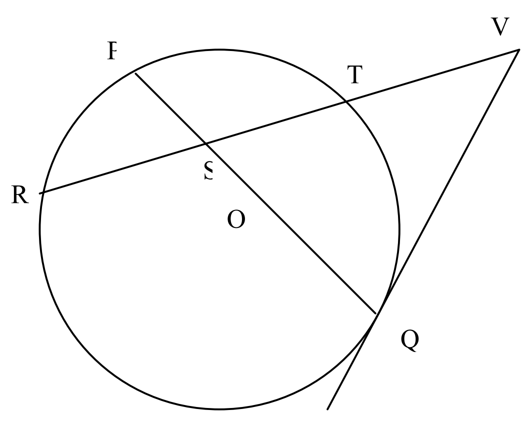
1. *ABCD* is a square with *AB* = 9, *EFGH* is a square in which *E* and *H* are trisection points of $\overline{AD}$, and *PQRS* is a square in which *R* is the midpoint of $\overline{BC}$ and P is the midpoint of $\overline{FG}$, as shown. If *X* is the area of *ABCD* and *Y* is the area of the shaded region, determine *Y*/*X*.



2. Trapezoid *ABCD* with $\overline{AB}$ parallel to $\overline{DC}$ and median $\overline{MN}$. If *AB* = 10, *DC* = 18, and *h* and *k* are the lengths of perpendiculars to $\overline{AC}$ drawn from *D* and *N*, determine $\frac{h}{k}$.



3. In circle O, if *PS* = n, *SQ* = n + 6, *RS* = n + 3, *ST* = n + 2, and *TV* = n + 1, determine the value of *VQ*. $\overline{VQ}$ is tangent to circle O at Q.



NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

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Round 4 – Algebra 2

1. ________________________

2. ________________________

3. ________________________

1. If $\log_5 2 = a$, $\log_5 3 = b$, and $\log_5 7 = c$, determine $\log_5 \frac{75}{16}$

2. Let f be a function such that $f(x) = xf(x-1)$ for all x and $f(3)$ is the arithmetic mean between $f(2)$ and $f(4)$. Determine the exact value of $\ln\left(5e^{f(3)}\right)$.

3. The reciprocal of a real number is 19/18 more than its square. What is the number?

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NEW ENGLAND PLAYOFFS – 2003

Round 5 – Analytic Geometry

1. ________________________

2. ________________________

3. ________________________

1. Line ℓ passes through the origin and divides square *ABCD* into two regions of equal area. Given *A*(7, 11), *B*(8, 11), *C*(8, 10) and *D*(7, 10), determine the slope of ℓ.

2. If the reflection of *A*(–1, 2) across $y = mx$ for $m > 0$ lies on the *x*–axis, determine *m*.

3. A circle passes through the vertex of $y = 4 - x^2$ and is tangent to the graph of $y = |x|$ at two points. Determine the radius of the smaller circle satisfying these conditions.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2003

Round 6 – Trig and Complex Numbers

1. ________________________

2. ________________________

3. ________________________

1. If $y = 479\cos(2003\pi x) + 821$ intersects $y = x$ at point P whose coordinates are integers, find the coordinates of P, expressed as an ordered pair.

2. Solve $\cos(\sin(\cos t)) = 1$ for $0 \le t < 2\pi$

3. A sine function, $y = D + A\sin(Bx + C)$, has a maximum y-value at $\left(\frac{3\pi}{8}, 5\right)$ and its next minimum y-value is at $\left(\frac{5\pi}{8}, 1\right)$. If A, B, C, and D are all positive, find the minimum value of $A + B + C + D$.

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

NEW ENGLAND PLAYOFFS – 2003

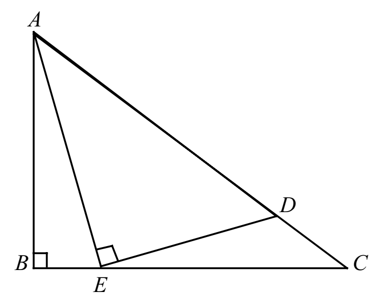
Team Round

1. ______________________ **4.** ____________________

2. ______________________ **5.** ____________________

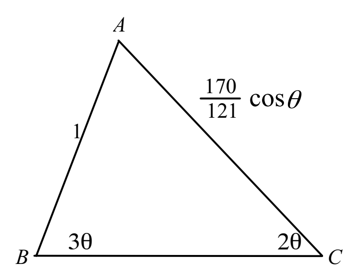
3. ______________________ **6.** ____________________

1. In right triangle ΔABC, $AB = 3$ and $BC = 4$. ΔDEA is inscribed in ΔABC such that $\Delta DEA \sim \Delta ABC$. If the perimeter of ΔDEA is m and the perimeter of ΔABC is n, determine m/n.

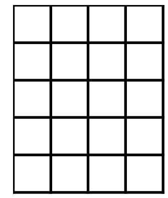


2. Let n be an <u>odd positive integer</u>. Consider $S = (0 + 1 + 2 + \ldots + n) + (1 + 2 + \ldots + n - 2 + n - 1) + (2 + 3 + 4 + \ldots + n - 3 + n - 2) + \ldots$ where each expression inside the parentheses has the outer terms of the previous expression deleted and the last expression has two terms. If $S = 3480$, determine n.

3. $AB = 1$, $AC = \dfrac{170}{121}\cos\theta$, $m\angle B = 3\theta$ and $m\angle C = 2\theta$. Determine the exact numerical value of $\cos\theta$.

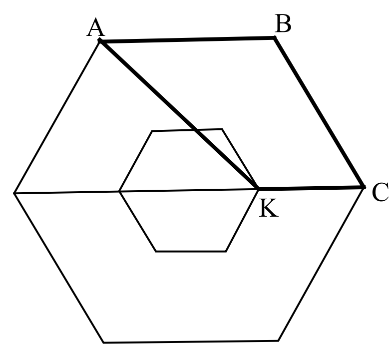


4. A square is chosen at random in both the first and second columns. Each is marked with an X and the mirror images of those squares across the vertical line separating the 2nd and 3rd columns are marked with an X in the third and fourth columns. For most combinations of marked squares there will be some 2 by 2 square regions in which no individual square is marked. Find the probability that the number of unmarked 2 by 2 regions is 7.



5. A constant m is selected with $0 < m < 1$. Find the sum of all solutions x to $\sin(x) = m$ if $0 \le x \le 2002$.

6. Shown are two concentric regular hexagons with parallel corresponding sides whose lengths are integers. The ratio of their areas is 4/25. Find the least area of trapezoid $ABCK$.



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NEW ENGLAND PLAYOFFS – 2003

Answer Sheet

Round 1

1. 64
2. -19
3. 17

Round 2

1. (a + 3b)(c – 4)
2. 3.6
3. $\frac{x}{y} = \frac{3+\sqrt{5}}{2}$

Round 3

1. $\frac{2}{3}$
2. $\frac{18}{5}$
3. $2\sqrt{42}$

Round 4

1. $2 + b - 4a$
2. ln 5
3. $\frac{2}{3}$

Round 5

1. $\frac{7}{5}$
2. $\frac{\sqrt{5}+1}{2}$
3. $4(\sqrt{2} - 1)$

Round 6

1. (1300, 1300)
2. $\frac{\pi}{2}, \frac{3\pi}{2}$
3. $9 + \pi$

Team

1. 25/32
2. 29
3. $\frac{11}{12}$
4. $\frac{4}{25}$
5. $2,003,001\pi$
6. $10\sqrt{3}$

NEW ENGLAND ASSOCIATION OF MATHEMATICS LEAGUES

PLAYOFFS – 2003 – Solutions Outline

Round 1 Arithmetic and Number Theory

1. 4♠3 = 4 + 9 =13; 5 ♦ 13 = 64

2. $\dfrac{2^2 \cdot 3^3 \cdot 5^5 \cdot 7^7}{(20 \cdot 30 \cdot 50 \cdot 70)^3} = \dfrac{2^2 \cdot 3^3 \cdot 5^5 \cdot 7^7}{(2 \cdot 3 \cdot 5 \cdot 7)^3 (10^4)^3} = \dfrac{2^2 \cdot 3^3 \cdot 5^5 \cdot 7^7}{2^{15} \cdot 3^3 \cdot 5^{15} \cdot 7^3} = 2^{-13} \cdot 3^0 \cdot 5^{-10} \cdot 7^4$. Thus,
   $a + b + c + d = -13 + 0 - 10 + 4 = -19$.

3. Clearly, $n = 2m^3$ and since $2 \cdot 17^3 = 9826$ and $2 \cdot 18^3 = 11664$, then $n = 2 \cdot 1^3, 2 \cdot 2^3, 2 \cdot 3^3, \ldots, 2 \cdot 17^3$ giving 17 values for n.

Round 2 Algebra 1

1. Multiplying and combining gives ac + 3bc – 4a – 12b = (a + 3b)(c – 4)

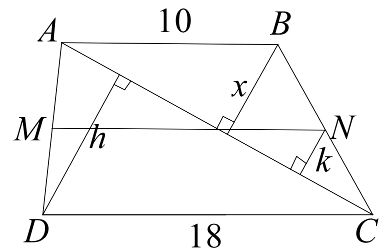
2. Let x be the length in meters. Then $\dfrac{x}{.9} = x + \dfrac{2}{5} \rightarrow x = \dfrac{18}{5} = 3.6$.

3. $\dfrac{x^3 - y^3}{x^3 - 3x^2y + 3xy^2 - y^3} = 4 \rightarrow \dfrac{(x-y)(x^2 + xy + y^2)}{(x-y)^3} = 4 \rightarrow$
   $x^2 + xy + y^2 = 4(x^2 - 2xy + y^2) \rightarrow x^2 - 3xy + y^2 = 0 \rightarrow \left(\dfrac{x}{y}\right)^2 - 3\left(\dfrac{x}{y}\right) + 1 = 0$. Thus,
   $\dfrac{x}{y} = \dfrac{3 \pm \sqrt{9 - 4 \cdot 1}}{2}$. Choose $\dfrac{x}{y} = \dfrac{3 + \sqrt{5}}{2}$.

Round 3 – Geometry

1. Let $AB = x$, making $EH = x/3$ and diagonal $PR = 2x/3$. The area of $EFGH$ is $\dfrac{x^2}{9}$ and the area of $PQRS$ is easily obtained using the area formula for a rhombus and equals $\dfrac{(2x/3)(2x/3)}{2} = \dfrac{4x^2}{9}$.
   Thus, $Y/X = \dfrac{x^2 - (x^2/9 + 4x^2/18)}{x^2} = \dfrac{2}{3}$.

2. Draw a perpendicular from B to $\overline{AC}$. Call its length x. (5/2)x. Since N is the midpoint of $\overline{BC}$, $\frac{k}{x}=\frac{1}{2}$. By alternate alterior angles, the right triangle with corresponding legs x and h are similar $\Rightarrow \frac{x}{h}=\frac{10}{18}$. Multiplying the two equations $\Rightarrow \frac{k}{h}=\frac{5}{18}$.



3. n(n + 6) = (n + 3)(n + 2) → n = 6. RS = 9, ST = 8 and TV = 7. (RV)(TV) = VQ^2.

Round 4 – Algebra 2

1. $\log_5 \frac{75}{16} = 2\log_5 5 + \log_5 3 - 4\log_5 2 = 2 + b - 2a$.

2. Since $f(3) = 3f(2)$ and $f(4) = 4f(3)$, then $f(3) = \frac{f(2)+f(4)}{2} = \frac{(1/3)f(3)+4f(3)}{2}$. Thus, $f(3) = \frac{13}{6}f(3) \rightarrow f(3) = 0$.

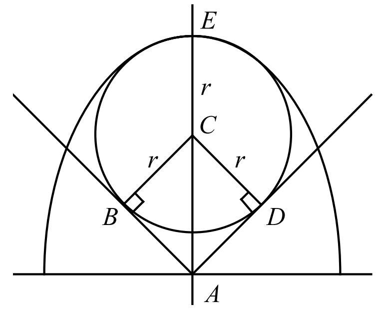
3. $\frac{1}{x} = x^2 + \frac{19}{18}$, $18x^3 + 19x - 18 = 0$, looking for root between 0 and 1. Use rational root theorem and synthetic division.

Round 5 – Analytic Geometry

1. A line passing through the center of a square bisects the square. The center of $ABCD$ is $M\left(\frac{15}{2},\frac{21}{2}\right)$ and the slope of ℓ is $\frac{21/2-0}{15/2-0} = \frac{7}{5}$.

2. The distance from the point (-1, 2) to the origin must = the distance from the origin to R, the reflection point. Hence we get $R\left(\sqrt{5},0\right)$. The slope of the segment connecting these points is $\frac{-2}{\sqrt{5}+1}$.

3. Since $m\angle BAD = 90°$, and $AB = AD$ while $CB = CD = r$, then $ABCD$ is a square. Thus, $CE = r$ and $AC = r\sqrt{2}$, so $CE + AC = r + r\sqrt{2} = 4 \rightarrow r = \dfrac{4}{\sqrt{2}+1} = 4(\sqrt{2}-1)$.



Round 6 – Trig and Complex Numbers

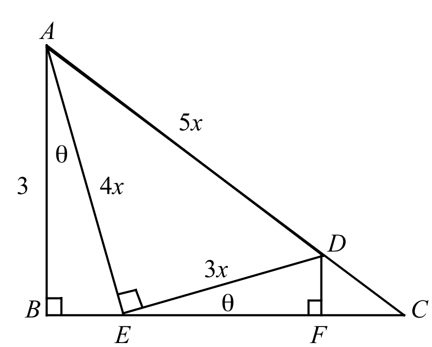
1. Since $479\cos(2003\pi x) + 821 = x$ for integer x, then $\cos(2003\pi x) = 1$ and $x = 479 + 821 = 1300$. The coordinates are (1300, 1300).

2. Let $x = \sin(\cos t)$. Then $\cos x = 1 \rightarrow x = 0 \pm 2\pi k \rightarrow \sin(\cos t) = 0 \pm 2\pi k$, but given that the range values of the sine function are restricted to between –1 and 1 inclusive, we have $\sin(\cos t) = 0$. Let $\cos t = y$. Then $\sin y = 0 \rightarrow y = 0 \pm \pi k$. Thus, $\cos t = 0 \pm \pi k$, but since $-1 \le \cos t \le 1$, $\cos t = 0 \rightarrow t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

3. Given $\left(\dfrac{3\pi}{8}, 5\right)$ and $\left(\dfrac{5\pi}{8}, 1\right)$, we know that the amplitude equals $\dfrac{5-1}{2} = 2$, so $A = 2$. The minimum value has been shifted upwards from –2 to 1, so $D = 3$. The period equals $2\left(\dfrac{5\pi}{8} - \dfrac{3\pi}{8}\right) = \dfrac{\pi}{2}$, and since $\dfrac{\pi}{2} = \dfrac{2\pi}{B}$, $B = 4$. Thus we have $y = 3 + 2\sin 4(x + \dfrac{C}{4})$. Point (0, 0) on $y = \sin x$ would be point $(-\dfrac{\pi}{4}, 3)$ on this graph so there is a phase shift to the left of $\dfrac{\pi}{4}$ making $\dfrac{C}{4} = \dfrac{\pi}{4}$ so $C = \pi$. Thus, $A + B + C + D = 2 + 4 + \pi + 3 = 9 + \pi$.

Team Round

1. Using $\Delta ABE \sim \Delta EFD$, $\cos\theta = \dfrac{3}{4x}$ and $\dfrac{EF}{3x}$ respectively. This makes $EF = 9/4$. Since $DC = 5 - 5x = 5(1-x)$ and $\Delta DFC \sim \Delta ABC$, then $DF = 3(1-x)$. Thus, $\left(\dfrac{9}{4}\right)^2 + \left(3(1-x)\right)^2 = (3x)^2$ making $x = \dfrac{25}{32}$. The ratio of perimeters equals $3x/3 = x = 25/32$.

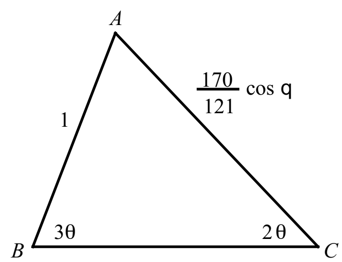


2. In S the first expression is an arithmetic progression with common difference $d = 1$ and $n + 1$ terms. The second has $d = 1$ and $n - 1$ terms, the third has $d = 1$ and $n - 3$ terms and so on until the last expression which has 2 terms . Thus, S equals

$$\left(\frac{0+n}{2}\right)(n+1)+\left(\frac{1+(n-1)}{2}\right)(n-1)+\left(\frac{2+(n-2)}{2}\right)(n-3)+....+\left(\frac{n-1}{2}+\frac{n+1}{2}\right)(n-(n-2))=$$

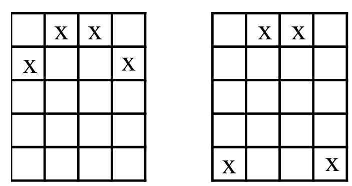
$n\left(\frac{n+1}{2}+\frac{n-1}{2}+\frac{n-3}{2}+...+1\right)$. The last expression inside ( ) = sum of the integers from 1 to $(n + 1)/2$, S equals $n \cdot \frac{1+\frac{n+1}{2}}{2} \cdot \frac{n+1}{2} = \frac{n(n+3)(n+1)}{8}$. Thus, n(n+1)(n+3)=8·3480 = 29·30·32, making $n = 29$.

3. Let $k = \frac{170}{121}$. By the Law of Sines,



$\frac{k\cos\theta}{\sin 3\theta} = \frac{1}{\sin 2\theta} \rightarrow k\cos\theta = \frac{\sin 3\theta}{\sin 2\theta} = \frac{3\sin\theta - 4\sin^3\theta}{2\sin\theta\cos\theta} = \frac{3-4\sin^2\theta}{2\cos\theta}$. Multiply both sides by $2\cos\theta$ and substitute $1 - \cos^2\theta$ for $\sin^2\theta$ giving $2k\cos^2\theta = 3 - 4 + 4\cos^2\theta$. This simplifies to $\cos^2\theta = \frac{1}{4-2k}$. Now $k = 170/121$ and obtain $\cos^2\theta = \frac{121}{144} \rightarrow \cos\theta = \frac{11}{12}$.

4. There are $5 \cdot 5 = 25$ different ways to select a square at random in the first two columns. Of those, the two shown in the diagram have exactly 7 clear 2 by 2 squares. For example, the left array has a 2 by 2 square bounded by the x's on the side and then the 3 by 4 section of squares at the bottom has six 2 by 2 that are free of x's. Each array can be turned upside down to give another array, making 4 arrays with 7 x-free 2 by 2's.



In the table below, the first row gives the number of 2 by 2 squares that are x–free and the second row gives the number of times that outcome occurs:

0	1	2	3	4	5	6	7	8	9
0	0	2	0	8	4	5	4	0	2

The answer is $\frac{4}{25}$.

5. Since m is non–negative, solutions lie in the first and second quadrant. They are $x, \pi - x, 2\pi + x, 3\pi - x, \ldots, 2000\pi + x, 2001\pi - x$. The x's cancel leaving

$$\pi(1 + 2 + 3 + \ldots + 2001) = \frac{(1 + 2001)(2001)\pi}{2} = 2,003,001\pi.$$

6. Since the ratio of areas is 4/25, the ratio of sides is 2/5. If $AB = 5$, then $FC = 10$. If $HJ = 2$, then $GK = 4$, giving $KC = 3$. Since $BC = 5$ and triangle BPC is a 30–60–90 triangle, then $PC = \frac{5\sqrt{3}}{2}$. Area of trapezoid $ABCK =$ $\frac{1}{2}(3 + 5)\frac{5\sqrt{3}}{2} = 10\sqrt{3}$.