IMLEM Meet #5
April, 2024

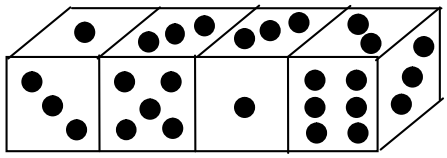
Intermediate Mathematics League of Eastern Massachusetts

Calculator Meet

• CALCULATORS: only *scientific calculators* allowed for meets #4 & #5) • Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.

Category 1
Mystery ***Calculator Meet***
Meet #5 - April, 2024

1) Four standard cubical dice, with faces numbered 1, 2, 3, 4, 5, and 6, are arranged as shown below with nine of the faces visible from this perspective. What is the total number of dots (called "pips") that are not visible in this view of the diagram, including the pips on the faces of the dice that touch one another?



2) If $\frac{A}{B} = 3$, then what is the value of $\frac{12B}{A}$?

3) In the multiplication to the right,

- **A, B, and C are different digits,**
- **A B is a two-digit number,**
- **1 C B is a three-digit number,**
- **no digit is zero, and**
- **A < B < C.**

What is the value of B ?

$$\begin{array}{r} A\ B \\ \times\quad B \\ \hline 1\ C\ B \end{array}$$

Answers

1) _____

2) _____

3) _____

Solutions to Category 1
Mystery
Meet #5 - April, 2024

Answers	
1)	57
2)	4
3)	5

1) The sum of the number of pips on any one die is 1 + 2 + 3 + 4 + 5 + 6 = 21. The sum on all four dice is 4 x 21, or 84. The sum of the pips shown is 1 + 3 + 3 + 5 + 3 + 1 + 2 + 6 + 3 = 27. The total number of pips on the faces hidden from view is 84 - 27, or 57.

2) If A / B = 3, then 3B = A. Then, substituting 3B for A, 12B / A = 12B / 3B = 4.

3) A bit of guessing and checking, along with some critical reasoning via process of elimination, can be expedient. Since the value of B occupies every position in the units column, the only possible values of B are 5 and 6. If B = 6, then for the product to fall between 100 and 200, A must be 2. Then 26 x 6 = 156. But then it would not be true that A < B < C. However, if B = 5, then for the product to fall between 100 and 200, A must be 3. Then 35 x 5 = 175. In addition, A < B < C. Therefore, B = 5.

Category 2
Geometry
Meet #5 - April, 2024

Calculator Meet

1) A rectangular box of facial tissues, measuring 4 inches by 6 inches by 9 inches contains, 124 tissues. Many such boxes fit perfectly into a shipping crate measuring 20 inches by 30 inches by 45 inches. How many tissues are in the crate?

2) The volume of a cube with a surface area of 29,400 square units is how many times the volume of a cube whose surface area is 294 square units?

3) An NCAA regulation basketball must have a diameter of from a minimum of 9.39 inches to a maximum of 9.55 inches. One basketball of the minimum size and one of maximum size are each packed into a separate cubical box so that each ball touches all six sides of its respective container. Each box has an amount of wasted space that lies outside the basketball but inside the box. How many more cubic inches of wasted space are there in the larger box than in the smaller box? Use $\pi \approx 3.14$. Round your answer to the nearest tenth of a cubic inch. (Note: Round *only* your final answer, as any intermediate rounding - that is, rounding during other parts of solving the problem - would yield answers increasingly farther from the more precise answer.)

Answers

1) _____

2) _____

3) _____

Solutions to Category 2
Geometry
Meet #5 - April, 2024

Answers
1) 15,500
2) 1,000
3) 20.5

1) Divide the volume of the crate by the volume of one tissue box: (20)(30)(45) / (4)(6)(9) = (5)(5)(5) = 125. So, the crate contains 125 tissue boxes, each containing 124 tissues. Multiply 125 by 124 = 15,500.

2) Solving by "brute force," which is time consuming but reliable, find the length of a side of each cube, find their respective volumes, then divide the larger volume by the smaller volume.

Area of one side of larger cube = 29,400 / 6 = 4900.
one side of larger cube = the square root of 4900 = 70.
Area of one side of smaller cube = 294 / 6 = 49.
one side of smaller cube = the square root of 49 = 7.
Volume of larger cube = (70)(70)(70) = 343,000
Volume of smaller cube = (7)(7)(7) = 343.
The ratio of the two volumes is 1000 : 1.
Therefore, the larger cube has 1000 times the volume of the smaller cube.

Some students may know that if two 3-D figures are similar, then the ratio of their areas is equal to the square of the ratio of any two corresponding lengths and the ratio of their volumes is equal to the cube of the ratio of any two corresponding lengths. This knowledge produces a much quicker calculation.

3) The radius of the larger ball is 9.55 / 2 = 4.775 inches.
The radius of the smaller ball is 9.39 / 2 = 4.695 inches.
One side of each box is as long as the diameter of the ball it contains.
Volume of box - volume of ball = volume of wasted space.

larger ball: $= D^3 - \frac{4}{3}\pi R^3 = (9.55)^3 - \frac{4}{3}(3.14)(4.775)^3 =$ **415.169.**

smaller ball: $= D^3 - \frac{4}{3}\pi R^3 = (9.39)^3 - \frac{4}{3}(3.14)(4.695)^3 =$ **394.6495.**

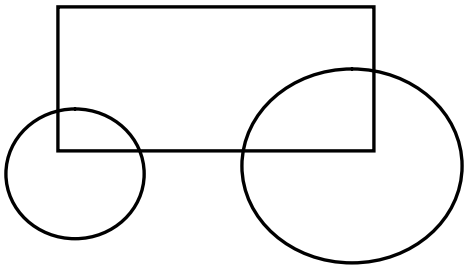
The difference in the wasted space in the two balls is 415.169 - 394.6495 = 20.5195 cubic inches.
Rounding to the nearest tenth yields 20.5.

Category 3
Number Theory
Meet #5 - April, 2024

Calculator Meet

1) Victoria has 83 coins in her pocket. Fifty-seven of the coins contain silver while 64 contain copper. Some of the coins are silver-copper alloys, which are mixtures of the two metals. How many of the coins are a silver-copper alloy?

2) Santana arranged some of the toys that his school donated to Toys for Tots into roped areas as shown. All 824 trucks are in the rectangle. The 342 orange toys are in the smaller circle while the 487 red toys are in the larger circle. There are 72 red trucks and 119 orange trucks. If 2265 toys were donated in all, then how many of the toys were outside of the areas that Santana had roped off?



3) Of the 460 kids at the school picnic, 30% ate a hot dog and 75% ate a hamburger. Of the kids who ate a hot dog, 65 of them also ate a hamburger. How many of the 460 kids ate neither a hot dog nor a hamburger?

Answers

1) _____

2) _____

3) _____

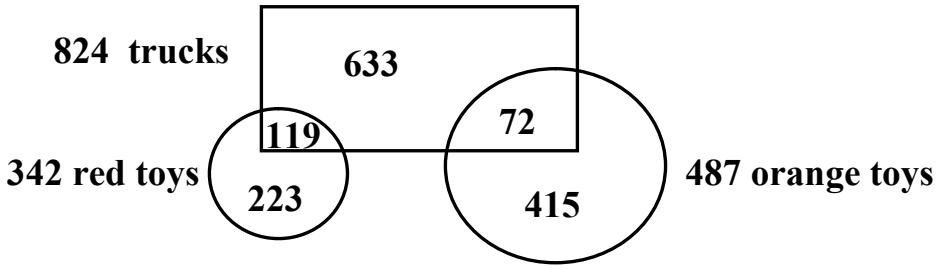
Solutions to Category 3
Number Theory
Meet #5 - April, 2024

Answers	
1)	38
2)	803
3)	42

1) Subtract the number of coins from the sum of silver + copper: (57 + 64) - 83 = 121 - 83 = 38.

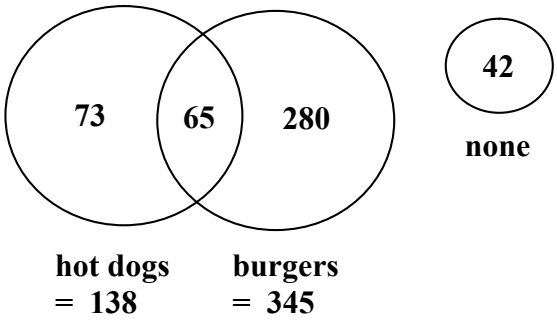
2) The numbers in this diagram reflect how many toys have the assigned characteristics:

2265 = total number of toys



The number of toys that lie outside the roped area is
2265 - (633 + 72 + 119 + 415 + 223)
= 2265 - 1462 = 803

3) The number of kids who ate hot dogs is 30% of 460, or 138. Those who ate hamburgers (burgers) is 75% of 460, or 280. The number who ate both a hot dog and a burger = 65, as given in the problem. To help fill in the blanks in the Venn diagram, those who ate just a hot dog = 138 - 65, or 73. Those who ate just a burger = 345 - 65, or 280. To calculate the number of kids who had neither a hot dog nor a burger, subtract the total number of kids who ate meat from the total number of kids who attended the picnic:
460 - (73 + 65 + 280)
= 460 - (418)
= 42
Therefore, 42 students ate neither a hot dog nor a hamburger.



Category 4 ***Calculator Meet***
Arithmetic
Meet #5 - April, 2024

1) A jar of colored candies has 17 red, 20 blue, and 23 yellow candies. Gabrielle chooses one piece of candy, at random. If the probability of choosing a blue piece of candy is the fraction A / B, in lowest terms, then what is the value of A + B ?

2) The probability, in a 4-child family, that there are three or more girls is expressed as a fraction C / D, reduced to lowest terms. What is the value of C + D ?

3) Frank has a bag of fruit chews containing the following flavors: 6 raspberry, 4 cherry, and 5 melon. If he chooses two at random, then what is the probability that they are the same flavor? If the answer is expressed as a fraction in lowest terms, then what is the sum of the numerator and denominator?

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 4
Arithmetic
Meet #5 - April, 2024

Answers	
1)	4
2)	21
3)	136

1) The total number of candies is 17 + 20 + 23, or 60. The probability of choosing a blue piece of candy is 20 / 60, or 1 / 3 when reduced to lowest terms. A + B = 1 + 3 = 4.

2) The fifth row of Pascal's Triangle can reveal some information about this problem, as presented to the right. The numbers 1 4 6 4 1 have a sum of 16. The probability that all four children are girls is 1 / 16. The probability of 3 girls and 1 boy is 4 / 16. The probability of 2 girls and 2 boys is 6 / 16. P(1 girl, 3 boys) is 4 / 16. P(0 girls, 4 boys) is 1 / 16. So, the probability of 3 or more girls is (1 + 4) / 16, or 5 / 16, already in lowest terms. So, C + D = 5 + 16 = 21.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

3) Both raspberry = (6 / 15)(5 / 14) = 30 / 210.
Both cherry = (4 / 15)(3 / 14) = 12 / 210.
Both melon = (5 / 15)(4 / 14) = 20 / 210.
Now add: 30 / 210 + 12 / 210 + 20 / 210 = 62 / 210 = 31 / 105, reduced to lowest terms.
The sum of the numerator and denominator is 31 + 105 = 136.

Category 5 *Calculator Meet*
Algebra
Meet #5 - April, 2024

1) There are two different values of X that make this equation true: What is the sum of those two values?

$$(X - 12)(X + 7) = 0$$

2) The quadratic equation $9x = 5 - 2x^2$ **has two solutions. Which solution has the lesser (smaller) value?**

3) A rocket is launched vertically from ground level at an initial velocity (starting speed) of 128 feet per second. For how many seconds is the rocket at least 112 feet above ground level? Use the quadratic equation $y = gt^2 + vt + h$

where g = - 16 feet/second/second, the constant of gravity at the surface of the Earth, t is the time in seconds that the rocket is in the air, v is the initial velocity, h is the initial height of the rocket in feet, and y is the height in feet of the rocket at any time t seconds.

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 5

Algebra

Meet #5 - April, 2024

Answers	
1)	5
2)	- 5
3)	6

1) If the product of two factors is zero, then either of the two factors has a value of zero. So, either X - 12 = 0 or X + 7 = 0. Then X = 12 or X = - 7 and their sum is 5.

2) Gather all terms onto one side of the equation, then use the strategy from solution #1.

$$9x = 5 - 2x^2$$
$$2x^2 + 9x - 5 = 0$$
$$(2x - 1)(x + 5) = 0$$
$$(2x - 1) = 0 \text{ or } (x + 5) = 0$$

X = 1/2 or X = - 5. The solution with the lesser (smaller) value is - 5.

3) Substitute: G = - 16; V = 128; Y = 112; H = 0.

$112 = (-16)(T^2) + 128T + 0$	**Use the substitutions listed above.**
$0 = -16(T^2) + 128T - 112$	**Subtract 112 from both sides.**
$0 = T^2 - 8T + 7$	**Divide both sides by - 16.**
$0 = (T - 1)(T - 7)$	**Factor.**

T = 1 or T = 7. Therefore, the rocket was at or above 112 feet above the ground from 1 second until 7 seconds into the flight, so the rocket was in flight for the difference 7 - 1, or 6 seconds.

Category 6
Team Round
Meet #5 - April, 2024

Each of the following six problems is worth <u>six</u> points.

1) If $\sqrt[3]{X\sqrt{X}} = 4$, then what is the value of X ?

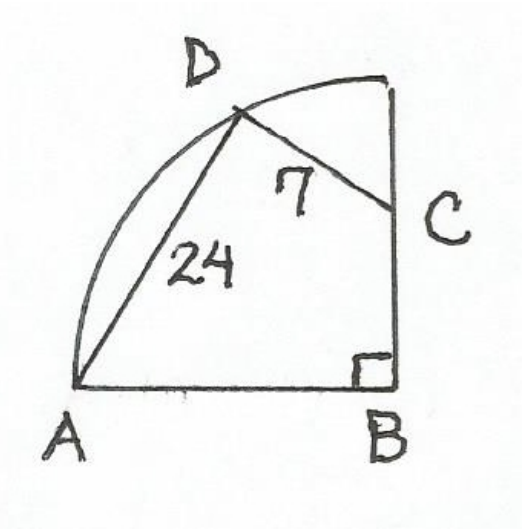
2) If $2^A + 2^B = 640$ then what is the value of A + B if A and B are positive integers?

3) The alphabet is written, over and over again, until 2024 letters are written in all. Each letter is assigned a positive integer value, such that A = 1, B = 2, C = 3, and so one, until Z = 26. What is the integer value of the 2024th letter?

4) Classrooms on one side of the corridor on the second floor at the Pentucket Regional Middle School are numbered with consecutive odd numbers from 401 through 499. Mr. Leonard's room is 3/4 of the way from room 413 to room 469 on the odd-numbered side of the corridor. What is Mr. Leonard's room number?

5) Elaine spent 2/3 of her money and then promptly spent 2/3 of what was left. If she finally ended up with $4 remaining, How many dollars did she have to start with?

6) Angle ADC is inscribed in a quarter-circle with center at point B. Angle ADC is a right angle. How many units long is BC if AD = 24 and DC = 7 ?



<u>ANSWERS</u>

1) _____

2) _____

3) _____

4) _____

5) _____

6) _____

Solutions to Category 6
Team Round
Meet #5 - April, 2024

ANSWERS	
1)	16
2)	16
3)	22
4)	455
5)	36
6)	15

1) Cubing both sides yields (X)(sqrt X) = 64.
A bit of number sense gives X = 16.

2) The powers of 2 that are less than 640 are 1, 2, 4, 8, 16, 32, 64, 128, 256, and 512. Find two whose sum is 640. Guessing and checking, along with some number sense, results in 128 + 512 = 640. 128 = 2^7. 512 = 2^9. A + B = 7 + 9 = 16.

3) Divide 2024 by 26 to get 77 full alphabets from A to Z, with remainder 22. The 22nd letter of the alphabet has a value of 22.

4) Three-fourths of (469 - 413) is (3/4)(56) = 42. So, 42 is added onto 413 to make room #455.

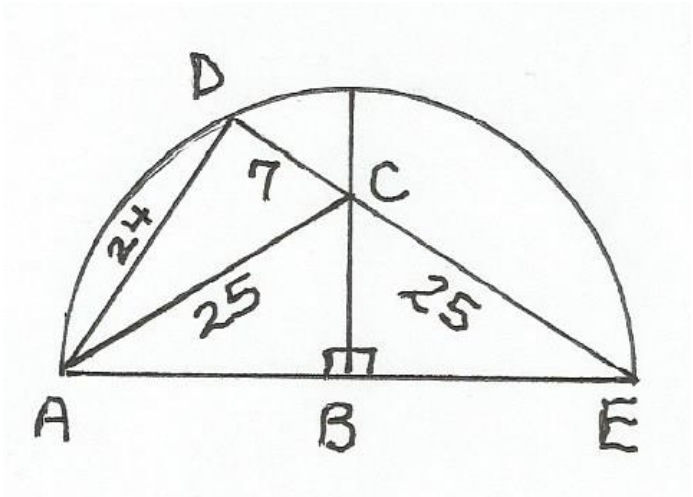
5) $X - (2/3)X - (2/3)(1/3)X = 4$
$1X - (2/3)X - (2/9)X = 4$
$1X - (8/9)X = 4$
$(1/9)X = 4$
$X = (4)(9)$
$X = 36$

Therefore, Elaine started with $36.

6) A couple of fundamental principles play a key role in this solution:
a) an angle inscribed in a semi-circle is a right angle, and
b) the Pythagorean Theorem.
Extend the arc to create a semi circle, with segment AB extended to a point that becomes the endpoint of a diameter. Extend segment DC to meet that endpoint. Also draw segment AC.

The remainder of the solution to problem #6 is on the next page.

Solution to #6, continued . . .



Since angle ADE is inscribed in a semi-circle, angle ADE is a right angle.
Employ the Pythagorean Theorem to find the length of AC:

$$7^2 + 24^2 = (AC)^2$$
$$49 + 576 = (AC)^2$$
$$625 = (AC)^2$$
$$25 = AC$$

Triangles ABC and EBC are congruent, so CE = 25.
Now add DC + CE = DE = 7 + 25 = 32.
Employ the Pythagorean Theorem to find the length of hypotenuse AE of right triangle ADE:

$$24^2 + 32^2 = (AE)^2$$
$$576 + 1024 = (AE)^2$$
$$1600 = (AE)^2$$
$$40 = AE$$

Half of diameter AE = 20.
Now focus on the small right triangle EBC and we're almost there.
Employ the Pythagorean Theorem:

$$20^2 + (BC)^2 = 25^2$$
$$400 + (BC)^2 = 625$$
$$(BC)^2 = 225$$
$$BC = 15.$$

Therefore, BC = 15 units.