IMLEM Meet #3
January, 2024

# Intermediate Mathematics League of Eastern Massachusetts


**CLUSTER COORDINATORS - A reminder to all students of some of the rules and of appropriate behavior during this meet:**

- **No calculators (or only scientific calculators allowed for meets #4, #5)**
- **Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.**

**Category 1**
**Mystery**
**Meet #3 - January, 2024**

**1) Half the students at one of our IMLEM member schools, the Jewish Community Day School (JCDS), study Algebra. One-fourth study Geometry. One-fifth study Arithmetic. Twenty students study Mathematics in Technology. All students take only one course. How many students are there at the JCDS?**

**2) A square number is the result of multiplying a whole number by itself. For example, 25 is a square number because it is the square of 5 such that (5) (5) = 25.**
**The difference of two positive whole numbers, M and N, is 6. The difference of their squares is 108. What is the value of the sum M + N ?**

**3) What is the only positive integer that has the following properties?**
- *** It is less than 100.**
- *** The sum of its digits is 7.**
- *** It is 6 less than a square number.**
- *** It is prime.**

**Answers**

**1)** _____

**2)** _____

**3)** _____

**Solutions to Category 1**
**Mystery**
**Meet #3 - January, 2024**

**1)** $1/2 + 1/4 + 1/5$
$= 10/20 + 5/20 + 4/20$
$= 19/20.$
So, 1/20 of the students are represented by the 20 students who take Mathematics in Technology. Then 20/20 of the students, or all the students at JCDS, = 400 students.

| Answers |
|---|
| 1) 400 |
| 2) 18 |
| 3) 43 |

**2)** Two numbers whose difference is 6 that may have square numbers in a range with a difference of 108 can be found by using number sense:

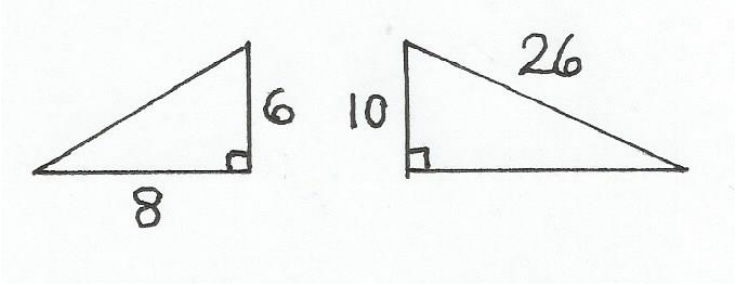
| Numbers with a difference of 6 | Their squares | Difference of squares |
|---|---|---|
| 4 and 10 | 16 and 100 | 84 |
| 5 and 11 | 25 and 121 | 96 |
| 6 and 12 | 36 and 144 | 108 |

So, $M + N = 6 + 12 = 18.$

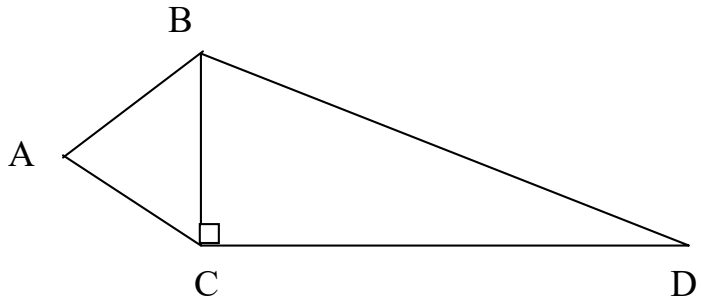
**3)** Of the four conditions, the second (sum of digits = 7) has a small number of possibilities: 16, 25, 34, 43, 52, 61, 70. The only one that is 6 less than a square is 43, as $43 + 6 = 49$, a square number. The number 43 is also prime.

**Category 2**
**Geometry**
**Meet #3 - January, 2024**

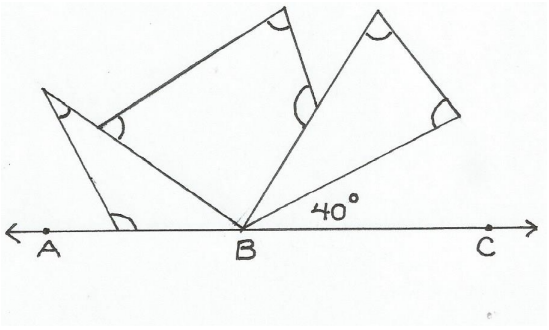
**1) Here are two right triangles with some side lengths given. Once you have computed the lengths of the two unlabelled sides, how many units are in the sum of the perimeters of the two triangles?**




**2) Triangle ABC is equilateral. AB = 10 inches. The area of right triangle BCD is 120 square inches. How many inches are in the perimeter of quadrilateral ABDC ?**




**3) In the figure below, How many degrees are in the sum of the angles marked with arcs? (note: The arcs do not indicate that the angles are congruent.)**




**Answers**

**1) _____ units**

**2) _____ inches**

**3) _____ degrees**

**Solutions to Category 2**
**Geometry**
**Meet #3 - January, 2024**

| Answers | |
|---|---|
| 1) | 84 |
| 2) | 70 |
| 3) | 580 |

**1) Students may recognize the triangles as 2x enlargements of the 3-4-5 and 5-12-13 "special" right triangles.**
**Double 5 to get 10 and double 12 to get 24.**
**Alternatively, utilize the Pythagorean Theorem twice:**

$6^2 + 8^2 = C^2$ $\qquad$ $10^2 + B^2 = 26^2$
$36 + 64 = C^2$ $\qquad$ $100 + B^2 = 676$
$100 = C^2$ $\qquad$ $B^2 = 576$
$10 = C$ $\qquad$ $B = 24$

**So, the sum of the perimeters is**
**$(6 + 8 + 10) + (10 + 24 + 26) = 24 + 60 = 84$ units.**

**2) If triangle ABC is equilateral, then each side = 10".**
**For triangle BCD, Area = (1/2)(base)(altitude)**
**120 = (1/2)(base)(10)**
**120 = (5)(base)**
**24 = base = CD**
**From the previous problem, the hypotenuse, BD, is 26 inches.**
**So, the perimeter of quadrilateral ABDC = AB + BD + CD + AC**
**= 10 + 26 + 24 + 10**
**= 70 inches**

**3) The four angles with point B as a common vertex have a sum of 180 degrees, including the 40 degree angle, so the sum of the remaining three unmarked angles is 180 - 40, or 140 degrees.**
**The two triangles each have an angle sum of 180 degrees while the quadrilateral has an angle sum of 360 degrees. The sum of all the angles of the three polygons is 180 + 180 + 360, or 720 degrees.**
**Subtract 140 from 720 to get 580 degrees, which is the sum of the angles marked with arcs.**

**Category 3**
**Number Theory**
**Meet #3 - January, 2024**

**1) The numeral 1234 is written in base 5. What is its base 10 value ?**

**2) In a certain numeration system with a certain base, the product (4) (4) = 31. In that base, what is the value of the sum 4 + 4 ?**

**3) Evaluate. If the final answer is expressed in scientific notation, in the form $A \times 10^B$, then what is the value of B ?**

$$\frac{72 \times 10^7}{1.2 \times 10^0} \times \frac{240 \times 10^{-6}}{0.09 \times 10^5} \div \frac{0.0008 \times 10^{-2}}{400 \times 10^{12}}$$

**Answers**

**1)** _____

**2)** _____

**3)** _____

**Solutions to Category 3**
**Number Theory**
**Meet #3 - January, 2024**

| Answers | |
|---|---|
| 1) | 194 |
| 2) | 13 |
| 3) | 20 |

**1) To convert 1234 base 5 to base 10:**
**= 4(1) + 3(5) + 2(25) + 1(125)**
**= 4 + 15 + 50 + 125**
**= 194.**

**2) The base would have to be 5 or greater, since 4 is used in the product.**
**(4)(4) = 16 in base 10. Base 5 accommodates this product as 3(5) + 1(1), or 31 base 5.**
**So, the sum 4 + 4, in base 10, is 8.**
**In base 5, that is 1(5) + 3(1), or 13 base 5.**

**3)** $$\frac{72 \times 10^{7}}{1.2 \times 10^{0}} \times \frac{240 \times 10^{-6}}{0.09 \times 10^{5}} \div \frac{0.0008 \times 10^{-2}}{400 \times 10^{12}}$$

$$= \frac{7.2 \times 10^{8}}{1.2 \times 10^{0}} \times \frac{2.4 \times 10^{-4}}{9 \times 10^{3}} \div \frac{8 \times 10^{-6}}{4 \times 10^{14}}$$

$$= \frac{7.2 \times 10^{8}}{1.2 \times 10^{0}} \times \frac{2.4 \times 10^{-4}}{9 \times 10^{3}} \times \frac{4 \times 10^{14}}{8 \times 10^{-6}}$$

$$= \frac{7.2 \times 2.4 \times 4 \times 10^{8} \times 10^{-4} \times 10^{14}}{1.2 \times 9 \times 8 \times 10^{0} \times 10^{3} \times 10^{-6}}$$

$$= 0.8 \times 10^{21}$$

$$= 8 \times 10^{20}$$

**Category 4**
**Arithmetic**
**Meet #3 - January, 2024**

**1) What is the value of** $2^5 + 5^2$ **?**

**2) On a standard keyboard, the notation A ^ B means** $_A{}^B$ **.**

**If 10 ^ W = 10,000**
**and 3 ^ X = 243**
**and 5 ^ Y = 5**
**and 7 ^ Z = 1**

**then what is the value of the product ( X ^ W ) ( Z ^ Y ) ?**

**3)** $A^{\frac{B}{C}}$ **means** $\sqrt[C]{A^B}$ **or, equivalently,** $\left(\sqrt[C]{A}\right)^B$

**Find the value of** $10{,}000^{\frac{3}{4}} - 64^{\frac{2}{3}} \cdot 32^{\frac{3}{5}}$

**ANSWERS**

**1)** _____

**2)** _____

**3)** _____

**Solutions to Category 4**
**Arithmetic**
**Meet #3 - January, 2024**

| Answers | |
|---|---|
| 1) | 57 |
| 2) | 0 |
| 3) | 872 |

1) $2^5 + 5^2 = (2)(2)(2)(2)(2) + (5)(5) = 32 + 25 = 57.$

2) 10 ^ W = 10,000 so W = 4.
3 ^ X = 243 so X = 5.
5 ^ Y = 5 so Y = 1.
7 ^ Z = 1 so Z = 0.

Then ( X ^ W ) ( Z ^ Y ) = ( 5 ^ 4 ) ( 0 ^ 1 )
= [(5)(5)(5)(5)] [(0)]
= [625] [0]
= 0.

A student who sees the "bigger picture" may fast forward to noting that the second factor in the expression has a value of zero, so there is no need to compute the value of 5 ^ 4, since (0)(any number) = 0.

3) $10{,}000^{\frac{3}{4}} - 64^{\frac{2}{3}} \cdot 32^{\frac{3}{5}} = \left(\sqrt[4]{10{,}000}\right)^3 - \left(\sqrt[3]{64}\right)^2 \cdot \left(\sqrt[5]{32}\right)^3$

$= 10^3 - 4^2 \times 2^3$

$= 1000 - 16 \times 8$

$= 1000 - 128$ (Multiply before subtracting!)

$= 872$

**Category 5**
**Algebra**
**Meet #3 - January, 2024**

**1) This question is about absolute value. What is the value of**

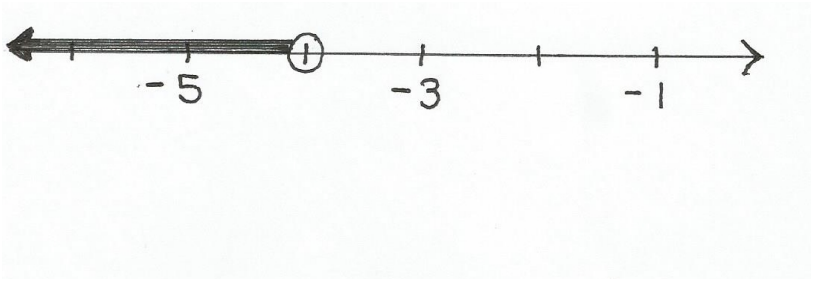
$$|23| + |-16| + |0|$$

**2) What is the product of all integers, N, that make the following absolute value inequality a true statement?**

$$\left|\frac{21}{N}\right| \geq 5$$

**3) For what value of W is the solution set ( in N ) of the following inequality given by the subsequent graph? Assume that the hatch marks on the graph represent integers that are equally spaced.**

$$4N - 2(3N - 5) + 5W > 33$$




**Answers**

1) _____

2) _____

3) _____

**Solutions to Category 5**
**Arithmetic**
**Meet #3 - January, 2024**

| Answers | |
|---|---|
| 1) | 39 |
| 2) | 576 |
| 3) | 3 |

**1)** $|23| + |-16| + |0| = 23 + 16 + 0 = 39.$

**2)** $\left|\frac{21}{N}\right| \geq 5$ So, integer values of N must fall between - 4 and 4, inclusive . . . but not zero, as division by zero is undefined. The set of possible values for N is { - 4, - 3, - 2, - 1, 1, 2, 3, 4 }. The product of the negative values is 24 and the product of the positive values is also 24. Finally, (24)(24) = 576.

**3)**
$$\begin{aligned} 4N - 2(3N - 5) + 5W &> 33 \\ 4N - 6N + 10 + 5W &> 33 \\ -2N + 10 + 5W &> 33 \\ -2N + 5W &> 23 \\ -2N &> 23 - 5W \\ N &< -11.5 + 2.5W \end{aligned}$$

And since, from the graph, N < 4, we must conclude that

$$\begin{aligned} -11.5 + 2.5W &= -4 \\ 2.5W &= 7.5 \\ W &= 3. \end{aligned}$$

**Category 6**
**Team Round**
**Meet #3 - January, 2024**

***Each of the following six problems is worth <u>six</u> points.***

**1) The length of a rectangle is increased by 40% and its width is increased by 10%, thus increasing its area by X%. What is the value of X ?**

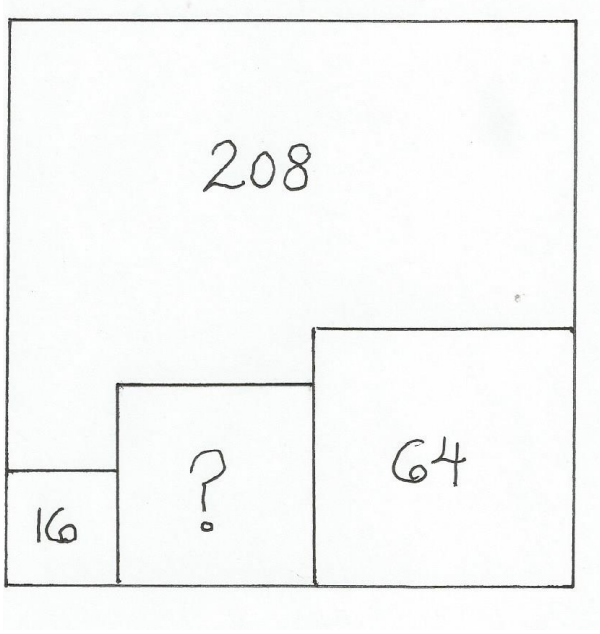
**2) There are 24 different 4-digit numbers, such that the digits of each number are 1, 2, 3, and 4, in some order. What is the sum of all 24 of the 4-digit numbers?**

**3) The number 25 is partitioned into three numbers whose ratio is 2 : 3 : 5. What is the value of the largest of the three numbers? Express your answer as a decimal.**

**4) Starting at midnight, Travis notices that the hands of a 12-hour clock form a right angle N number of times over the next 24 hours. What is the value of N ?**

**5) Notice that the whole numbers 13 and 31 are both prime and have the same digits, but reversed. There are three more pairs of two-digit numbers with this property. What is the sum of those six two-digit numbers?**

**6) A large square contains three smaller squares and an octagon. Two of the squares have areas of 16 and 64 square units, respectively, and the area of the octagon is 208 square units, as shown below. How many square units are in the square indicated by the question mark?**




**ANSWERS**

**1)** _____

**2)** _____

**3)** _____

**4)** _____

**5)** _____

**6)** _____

**Solutions to Category 6**
**Team Round**
**Meet #3 - January, 2024**

**1)** The new larger rectangle measures 1.4L by 1.1W with an area of ( 1.4L) (1.1W ) = 1.54LW, thus increasing the area of the original rectangle by 54%. Therefore, X = 54.

**2)** Picturing the tall addition column, there are six of each digit in each of the four columns. Each column has a value of $6(1) + 6(2) + 6(3) + 6(4) = 6(1 + 2 + 3 + 4) = 6(10) = 60$.
In the units, or ones column, put down the zero and carry the 6. The sum in the tens column is 66, so put down the 6 and carry the 6. Same for the hundreds and thousands columns. Total = 66,660.

**3)** Let X be the scaling factor so that $2X + 3X + 5X = 25$, $10X = 25$, and $X = 2.5$. Then the largest number's value is (5)(2.5), or 12.5.

**ANSWERS**

1) 54
2) 66,660
3) 12.5
4) 44
5) 374
6) 36

**4)** During each of the 24 hours, the hands of the clock make a right angle twice . . . for example, near 1:20 and 1:55. One might conclude that the answer is (2)(24), or 48. However, in the 2:00-3:00 range, this phenomenon only occurs once. Same with the 8:00-9:00 range. So, subtract two right angles for each 12-hour period, or subtract four right angles for a 24-hour period, and the answer is 44.

**5)** The three additional pairs of primes are 17 & 71, 37 & 73, and 79 & 97. The sum of those six numbers is $17 + 71 + 37 + 73 + 79 + 97 = 374$.

**6)** The 16-square has a side length of 4 and the 64-square has a side length of 8. Let one side of the ?-square be X. Its area is $X^2$. Then one side of the largest square is $4 + 8 + X$, or $X + 12$. Its area is $(X + 12)(X + 12)$, or $X^2 + 24X + 144$. The area of the largest square

**The remainder of the solution to #6 is on the next page.**

**can also be represented by the sum of its parts, or 208 + 16 + 64 + X^2.**

**So, X^2 + 24X + 144 = 208 + 16 + 64 + X^2**
**X^2 + 24X + 144 = 288 + X^2**
**Now subtract X^2 from both sides, resulting in**
**24X + 144 = 288**
**24X = 144**
**X = 6.**

**Finally, the area of the square represented by ? is (6)(6), or 36 square units.**