Intermediate Mathematics League of Eastern Massachusetts

Calculator Meet

• CALCULATORS: only *scientific calculators* **allowed for meets #4 & #5) • Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.**

1) What is the absolute value of the product of all the integers from - 6 to
   3, inclusive?

2) On April 13, 1987, Rick was 8 times as old as Pat while Kelly was
   3 times as old as Pat. Rick is 20 years older than Kelly. How old was
   Pat on April 13 in the year 2023?

3) Two cars are travelling in the same direction. The car that is in the
   lead is going 48 miles per hour while the second car is going 60 miles
   per hour. How many miles apart are the two cars 45 minutes before
   the faster car catches up to the slow car?

Answers

1) _____

2) _____

3) _____

1) The consecutive integers in this range includes the integer zero. Any product that includes zero as a factor produces a product of zero.

2) A student could use "guess and check" or "make a table" as strategies. Here is an algebraic solution:
Let X = Pat's age
     3X = Kelly's age and
     8X = Rick's age.
Then 8X = 3X + 20
        5X = 20
          X = 4.
Then X + 36 = 4 + 36 = 40,   as there are 36 years between the two dates.
So, Pat was 40 years old on April 13, 2023.

3) In one hour, the faster car goes 12 miles farther than the slower car, the difference in their two rates of speed in miles per hour. So, 45 minutes is 3/4 of an hour, then 3/4 of 12 is 9 miles, the distance apart the cars are 45 minutes before the faster car catches up to the slower car.

Answers	
1)	0
2)	40
3)	9

1) A common brick, in the shape of a rectangular solid, measures 5 by 9 by 23 centimeters. How many square centimeters are in its total surface area?

2) The edge of one cube is 3 centimeters longer than the edge of another cube. The difference between their volumes is 1,413 cubic centimeters. How many centimeters are in the length of the larger cube?

3) A water tank in the shape of a cylinder has an inside diameter of 24 feet and a height of 87 feet. A cubic foot of water is 7.5 gallons. A rock, in the shape of a sphere with a diameter of 8 feet, is dropped into a full tank of water and sinks to the bottom of the tank. To the nearest whole number of gallons, how many gallons of water remain in the tank ? Use $\pi \approx 3.14.$

Answers

1) _____

2) _____

3) _____

Solutions to Category 2
Geometry
Meet #5 - April, 2023

1) **Surface area** = (2)(5)(9) + (2)(5)(23) + (2)(9)(23)
                    = 90 + 230 + 414
                    = 734 square centimeters

2) The volume of a cube = (one edge)^3
    One strategy is to compute the volumes of cubes of
    various side lengths and compute the differences between
    the volumes of cubes with side lengths of integers that
    are three apart.

Side length	Volume	Difference to volume of cube with side length 3 less
5	125	- - -
6	216	- - -
7	343	- - -
8	512	387 = 512 - 125
9	729	513 = 729 - 216
10	1000	657 = 1000 - 343
11	1331	819 = 1331 - 512
12	1728	999 = 1728 - 729
13	2197	1197 = 2197 - 1000
14	2744	1413 = 2744 - 1331

So, there are 14 centimeters in the length of the larger cube.

3) Subtract the volume of the spherical rock from the volume of the cylindrical
    tank, then multiple the number of cubic feet by 7.5, the number of gallons in
    a cubic foot.

    (7.5) [(pi)(radius^2)(height) - (4/3)(pi)(radius^3)]
    = (7.5) [(3.14)(12^2)(87) - (4/3)(3.14)(4^3)]
    = (7.5) [39,337.92 - 267.9466]
    = (7.5) [39,069.974]
    = 293,024.8 which rounds to 293,025 gallons.

1) A = { positive multiples of 5 that are less than 200 }
   B = { positive multiples of 7 that are less than 200 }
   How many numbers are in both sets A and B (the intersection of
   sets A and B) ?

2) Of the 287 8th graders at the Northborough Melican Middle School,
   209 are less than six feet tall,
   156 have dark hair,
   123 are less than six feet tall and have dark hair.
   How many students are at least six feet tall and do not have dark hair?

3) M = { 2, 3, 5, 7, 11, 13 }
   A = { 9, 18, 27, 36, 45, 54, 63 }
   T = { 6, 12, 18, 24, 30, 36, 42, 48, 54, 60 }
   H = { 3, 5, 7, 9 }

   If P = $(M \cap H) \cup (A \cap T)$ then what is the product of all

   the numbers in set P ?   (Translation: P is the union of the
   intersection of sets M and H and the intersection of sets A and T.)

Answers

1) _____

2) _____

3) _____

Solutions to Category 3
Number Theory
Meet #5 - April, 2023

1) **The numbers in both sets A and B are the common multiples of 5 and 7 that are less than 200, or the set { 35, 70, 105, 140, 175 }. There are five numbers in this set.**

2) **Add the first two sets and then subtract their intersection: 209 + 156 - 123 = 242. This is the number of students who are less than six feet tall and have dark hair. The question asks for the opposite - the number who are at least six feet tall and do not have dark hair. So, subtract 242 from 287 = 45.**

3) **The intersection of M and H is {3, 5, 7 }. The intersection of A and T is { 18, 36, 54 } The union of the two intersections is { 3, 5, 7, 18, 36, 54 } The product of the six numbers in the intersection is**
     (3)(5)(7)(18)(36)(54) = 3,674,160

1) Heidi wrote the number 1 on one slip of paper. She wrote the number 2 on two slips of paper, the number 3 on three slips of paper, the number 4 on four slips of paper, the number 5 on five slips of paper, and the number 6 on six slips of paper. All of the slips of paper were placed into a box. Sally withdrew one slip of paper from the box. The probability, written as a fraction in lowest terms in lowest terms, that the number Sally withdrew is an odd number is X / Y. What is the sum of X + Y ?

2) The probability that Evan will successfully make a basket during a basketball game is 2 / 5. If the probability that he misses a basket on three consecutive shots is A / B, written as a fraction in lowest terms, then what is the product of AB ?

3) A vendor is selling balloons at the parade. She has 7 red, 4 green, and 5 blue balloons remaining. Cagney and Lacey each select a balloon without one seeing what the other has chosen. The probability, written as a fraction in lowest terms, that Cagney and Lacey each select a balloon of the same color is K / M. What is the sum of K + M ?

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 4
Arithmetic
Meet #5 - April, 2023

1) **Total # of pieces of paper: $1 + 2 + 3 + 4 + 5 + 6$**
   $= 21$. The total number of odd numbers is
   $1 + 3 + 5 = 9$. The probability that Sally
   withdrew an odd number is $9 / 21$, or $3 / 7$
   in lowest terms. Finally, $X + Y = 3 + 7 = 10$.

2) **If the probability of making a basket is $2 / 5$,**
   then the probability of missing a shot is $3 / 5$.
   The probability of missing on three consecutive
   shots is $(3/5)(3/5)(3/5)$, or $27 / 125$. The numbers 27 and 125 are
   relatively, prime, so the fraction $27 / 125$ is already in lowest terms.
   So, the product of $AB = (27)(125) = 3,375$.

3) **This problem is a "probability without replacement" problem, so that**
   once a balloon is selected, there is one fewer balloon among those
   remaining. There is a total of $7 + 4 + 5$, or 16 balloons.
   The probability of Cagney and Lacey each selecting the same color is
   $$\text{(red, red)} + \text{(green, green)} + \text{(blue, blue)}$$
   $$= (7 / 16)(6 / 15) + (4 / 16)(3 / 15) + (5 / 16)(4 / 15)$$
   $$= 42 / 240 + 12 / 240 + 20 / 240$$
   $$= (42 + 12 + 20) / 240$$
   $$= 74 / 240$$
   $$= 37 / 120.$$
   Finally, the sum of $K + M = 37 + 120 = 157$.

1) There are two possible values of X that make this equation true:

$$(X - 8)(X + 5) = 0$$

What is the sum of those two values?

2) In a right triangle, the length of one leg is 2 centimeters shorter than twice the length of the other leg. If the area of the triangle is 132 square centimeters, then how many centimeters are in the length of the longer leg?

3) The quadratic equation $Y = aX^2 + bX + c$, when graphed, is a parabola that passes through the points (3, 7) and (- 5, 15) and has a Y-intercept of (0, - 20), as shown.
What is the value of   a + b + c ?

(- 5, 15)

(3, 7)

(0, - 20)

ANSWERS

1) _____

2) _____

3) _____
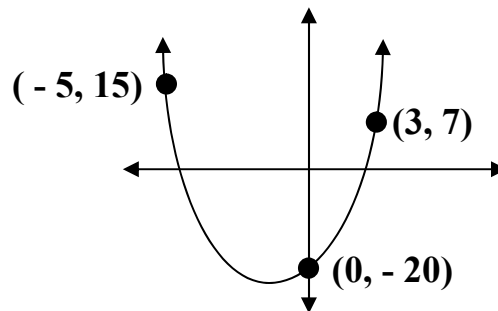
Solutions to Category 5
Algebra
Meet #5 - April, 2023

Answers

1)   3

2)   22

3)   - 15

1) If the product of two factors is zero, then
either of the two factors has a value of zero.
So, either X - 8 = 0   or   X + 5 = 0.
Then X = 8 or X = - 5 and their sum is 3.

2) Let N = the length of the shorter leg.
Then 2N - 2 = the length of the longer leg.
The area of the right triangle
= (1/2)(base)(altitude)

So, 132 = (1/2)(N)(2N - 2)   . . .   132 = (1/2)(2N^2 - 2N)
264 = 2N^2 - 2N   . . .   132 = N^2 - N
0 = N^2 - N - 132 . . . 0 = (N - 12)(N + 11) . . . N = 12 or - 11.
Since the length of a side of a triangle can only be positive, then the
shorter leg is 12 cm and the longer leg is 2(12) - 2, or 22 cm.

2) One possible strategy:
1: substitute the X and Y coordinates of each of the known points
into the general quadratic equation,
2: solve the resulting system to find the values of a, b, and c, then
3: find the sum a + b + c.

for (3, 7): $7 = a\left(3^2\right) + b(3) + c$ . . . or, simplified, $7 = 9a + 3b + c$

for (- 5, 15): $15 = a\left((-5)^2\right) + b(-5) + c$ . . . or, simplified, $15 = 25a - 5b + c$

for (0, - 20): $-20 = a\left((0)^2\right) + b(0) + c$ . . . or, simplified, $-20 = c$

Now substitute - 20 for c in the first two equations, yielding
7 = 9a + 3b - 20   and   15 = 25a - 5b - 20
Simplifying: 27 = 9a + 3b   and   35 = 25a - 5b
Divide both sides of the first equation by 3 and both sides of the
second equation by 5, yielding: 9 = 3a + b   and   7 = 5a - b. Adding
the two equations yields: 16 = 8a, so, a = 2 and then b = 3. So,
a + b + c = - 15.

Each of the following nine problems is worth __four__ points.

1) A small square is drawn somewhere inside a larger square. The interior of the small square is colored red. The area outside the red square but inside the larger square is colored blue. The red area = one-eighth of the blue area. If the ratio of the length of one side of the larger square to the length of one side of the smaller square is T:U, and if that ratio is in lowest terms, then what is the value of T ?

2) The average (arithmetic mean) weight of five people is 150.4 pounds. No person weighs less than 130 pounds. The difference in the weights of any two people is at least 5 pounds. What is the greatest number of pounds that any one of the people can weigh?

3) Let the lengths of the sides of a triangle be represented by X + 3, 2X - 3, and 3X - 5. If the perimeter of the triangle is 25 units, then how many units long is the shortest side?

4) Square ABCD is constructed on a Cartesian coordinate plane. Vertex A is at (- 6 - 6) while vertex C is at (13, 13). How many square units are in the area of square ABCD?

5) The average (arithmetic mean) of D, E, F, G, and H is 67. The average of D, E, and H is 49. What is the average of F and G?

6) Jayson scored on 80% of his foul shots in basketball game while Jaylen scored on 75% of his foul shots. They each scored on 12 shots. How many total shots did they take?

7) The area of a square is 7 square inches. If the length of each side is tripled, then how many square inches are in the area of the new square?

QUESTIONS #8 and #9 ARE ON THE NEXT PAGE.

__ANSWERS__

1) _____

2) _____

3) _____

4) _____

5) _____

6) _____

7) _____

8) _____

9) _____

8) The rectangular top surface of a cement block in the shape of a rectangular solid measures 19 by 37 centimeters. The block is 23 centimeters high. There are two identical rectangular holes that pass through the block from top to bottom, each measuring 11 by 15 centimeters. How many cubic centimeters of cement were required to make the block?

9) More than 20 students but fewer than 50 students paid a total of $ 731 for theater tickets. If each ticket costs the same whole number of dollars, then how many students are in the group?

Solutions to Category 6
Team Round
Meet #5 - April, 2023

<table>
<tr><td colspan="2" align="center">__ANSWERS__</td></tr>
<tr><td>1)</td><td>3</td></tr>
<tr><td>2)</td><td>202</td></tr>
<tr><td>3)</td><td>7</td></tr>
<tr><td>4)</td><td>361</td></tr>
<tr><td>5)</td><td>94</td></tr>
<tr><td>6)</td><td>31</td></tr>
<tr><td>7)</td><td>63</td></tr>
<tr><td>8)</td><td>8,579</td></tr>
<tr><td>9)</td><td>43</td></tr>
</table>

1) The larger square has 9 times the area of the smaller square, so the ratio of one side of the larger square to one side of the smaller square is the square root of 9, or 3.

2) The total weight of all the people is (5)(150.4), or 752 pounds. If we are looking for greatest weight, then we must minimize the other four and make those numbers five apart. So, the lightest four weigh 130, 135, 140, and 145 pounds, respectively. The total of these four is 550 pounds, so the heaviest person weighs 752 - 550, or 202 pounds.

3) $(X + 3) + (2X - 3) + (3X - 5) = 25$
$$6X - 5 = 25$$
$$6X = 30$$
$X = 5$, so $X + 3 = 8$, $2X - 3 = 7$, and $3x - 5 = 10$. Finally, the shortest side is $2X - 3$, or 7 units.

4) The length of each side is the difference 13 - (- 6), or 13 + 6, or 19 units. The area of the square is (19)(19), or 361 square units.

5) If the average of five numbers is 67, then their sum is (5)(67), or 335. If the average of the next three numbers is 49, then their sum is (3)(49), or 147. The difference of these two sums is 335 - 147, or 188. So, the average of the remaining two numbers is 188 / 2, or 94.

6) Twelve is 80% of Jayson's total number of shots, so his total number of shots is 12 / 0.8, or 15 shots. Twelve is 75% of Jaylen's total number of shots, so his total number of shots is 12 / 0.75, or 16 shots. Adding the two totals: 15 + 16 = 31 total shots.

THE SOLUTIONS FOR # 7 - 9 ARE ON THE NEXT PAGE.

7) The ratio of the areas of two squares is equal to the square of the
ratio of their corresponding sides. So, the area of the larger square is
3^2, or 9 times as great as the area of the smaller square.
Then (9)(7) = 63 square inches.

8) The volume of the block is equal to the volume of the block,
including the holes, minus the volume of the holes,
= (19)(37)(23) - (2)(11)(15)(23)
= 16,169 - 7,590
= 8,579   cubic centimeters.

9) Locate a prime number between 20 and 50 that is a factor of 731.
Since there are just a few such prime numbers, try dividing 731 by
23, 29, 31, 37, 41, 43, and 47 until a whole number quotient is the result.
Here is the only one that works: 731 / 43 = 17.
Since the number of students must be between 20 and 50, the
number of students must be 43 and the price for each ticket must
be $ 17.