

IMLEM Meet #3  
January, 2023

# Intermediate Mathematics League of Eastern Massachusetts



**CLUSTER COORDINATORS** - A reminder to all students of some of the rules and of appropriate behavior during this meet:

- No calculators (or only scientific calculators allowed for meets #4, #5)
- Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.

## Category 1

### Mystery

### Meet #3 - January, 2023



- 1) When I was 13 years old, my older sister was twice my age. I am now 42 years old. How many years old is my sister?
- 2) A number line of integers was drawn onto the floor of a gymnasium. A puppy was sitting on the negative number  $-13$  and began walking toward the positive numbers. She was given a treat at a point that was  $\frac{5}{8}$  of the way from  $-13$  to  $43$ . On what number was the puppy given a treat?
- 3) The sum of two positive integers is 15. What is the smallest possible sum of their reciprocals? If that sum is a fraction in lowest terms, then what is the sum of its numerator and denominator?

### Answers

1) \_\_\_\_\_

2) \_\_\_\_\_

3) \_\_\_\_\_

**Solutions to Category 1  
Mystery  
Meet #3 - January, 2023**

1) The sister was twice 13, or 26 years old. The difference in their ages is 13 years and will continue to be so. If I am now 42 years old, then my sister is 13 years older, or  $42 + 13$ , or 55 years old.

2) The difference between 43 and - 13 is  $43 - (- 13)$ , or 56.  $\frac{5}{8}$  of that difference is  $(\frac{5}{8})(56)$ , or 35. Add 35 to - 13 to get 22. The puppy is fed her treat on the number 22.

3) The smallest sum of reciprocals occurs when the two addends of that sum are closest in value. Students may discover that by experimenting if they don't already know.

$7 + 8 = 15$  and the sum of the reciprocals is  $\frac{1}{7} + \frac{1}{8}$  or  $\frac{15}{56}$ .

$6 + 9 = 15$  and the sum of the reciprocals is  $\frac{1}{6} + \frac{1}{9}$  or  $\frac{15}{54}$  which is larger than the previous sum.

$5 + 10 = 15$  and the sum of the reciprocals is  $\frac{1}{5} + \frac{1}{10}$  or  $\frac{15}{50}$  which is larger than the previous sums.

And so on and so one.

Therefore, the smallest possible sum of reciprocals is  $\frac{15}{56}$ , a fraction in lowest terms. The sum of the numerator and denominator is  $15 + 56$ , or 71.

**Answers**

1) 55

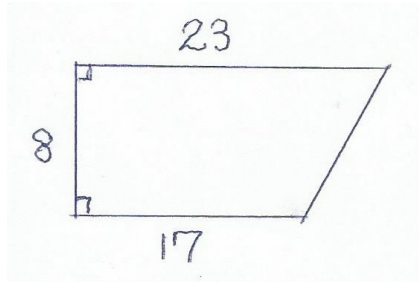
2) 22

3) 71

**Category 2**  
**Geometry**  
**Meet #3 - January, 2023**

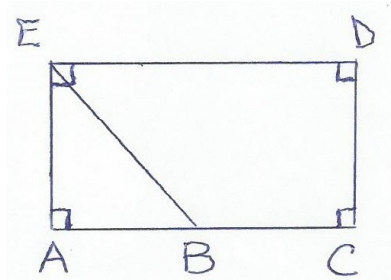


1)



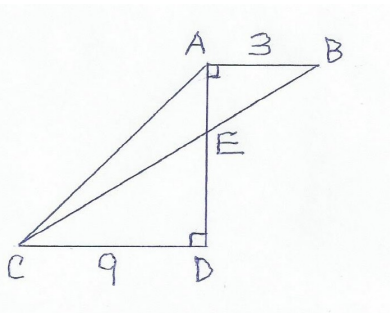
The measures of three segments of a right trapezoid are given, as marked on the figure. How many units long is the remaining side?

2)



$AB = 12$  centimeters.  
 $DE = 27$  centimeters.  
 The area of triangle  $ABE$  is 30 square centimeters.  
 How many centimeters are in the perimeter of right trapezoid  $BCDE$ ?

3)



$AD = 4$  inches.  
 $AB = 3$  inches.  
 $CD = 9$  inches.  
 How many square inches are in the area of triangle  $AEC$  ?

**Answers**

- 1) \_\_\_\_\_ units  
 2) \_\_\_\_\_ cm  
 3) \_\_\_\_\_ sq. in.

**Solutions to Category 2**  
**Geometry**  
**Meet #3 - January, 2023**

<u>Answers</u>	
1)	10
2)	60
3)	4.5

- 1) The difference between the two horizontal lengths is 23 - 17, or 6. draw a perpendicular from A to the upper horizontal side. That side is 8 and is the altitude of a right triangle whose short side is 6. Employ the Pythagorean Theorem to find the length of side AB:

$$\begin{aligned} 6^2 + 8^2 &= (AB)^2 \\ 36 + 64 &= (AB)^2 \\ 100 &= (AB)^2 \\ 10 &= AB \end{aligned}$$

- 2) The area of triangle ABE is 30 square centimeters, so work backwards to find the length of altitude AE: Area = (1/2)(base)(altitude)

$$\begin{aligned} 30 &= (1/2)(12)(AE) \\ 30 &= (6)(AE) \\ 5 &= AE \end{aligned}$$

ACDE is a rectangle, given the four right angles, so that AC = ED.

Then BC = DE - AB, BC = 27 - 12 = 15.

DC = AE because opposite sides of a rectangle have the same measure.

So, DC = 5. Employ the Pythagorean Theorem to find the length of EB:

$$\begin{aligned} (AB)^2 + (AE)^2 &= (EB)^2 \\ 12^2 + 5^2 &= (EB)^2 \\ 144 + 25 &= (EB)^2 \\ 169 &= (EB)^2 \\ 13 &= EB \end{aligned}$$

Then the perimeter of BCDE = BC + DC + DE + EB  
= 15 + 5 + 27 + 13  
= 60 centimeters.

- 3) Triangles ABE and CDE are similar via A-A, so their corresponding sides are proportional. Then AE : 3 = ED : CD

$$\begin{aligned} X : 3 &= (4 - X) : 9 \\ 9X &= 3(4 - X) \\ 9X &= 12 - 3X \\ 12X &= 12 \\ X &= 1. \end{aligned}$$

The area of triangle AEC = area of triangle ABC - area of triangle ABE  
= (1/2)(3)(4) - (1/2)(3)(1) = 6 - 1.5 = 4.5.

**Category 3**  
**Number Theory**  
**Meet #3 - January, 2023**



- 1) The numeral 10110 is written in base 3. What is its base 10 value ?
  
  
  
  
  
  
  
  
  
  
- 2) In the 1992 presidential election, Bob Dole received some write-in votes. Bill Clinton got 5000 times as many votes with 67,800,000 votes. How many votes did Bob Dole receive? In scientific notation, if your answer is in the of the form  $A \times 10^N$ , then what is the value of  $A + N$  ? Express your answer as a decimal.
  
  
  
  
  
  
  
  
  
  
- 3) Four numbers are written in different bases as follows:

Base   Number

2	11
3	222
4	3333
5	44444

What is the value of the sum of these four numbers if it is written in base 6 ?

Answers

1) \_\_\_\_\_

2) \_\_\_\_\_

3) \_\_\_\_\_

**Solutions to Category 3**  
**Number Theory**  
**Meet #3 - January, 2023**

1) To convert 10110 base 3 to base 10:  
 $= 0(1) + 1(3) + 1(9) + 0(27) + 1(81)$   
 $= 0 + 3 + 9 + 0 + 81$   
 $= 93.$

2)  $6.78 \times 10^7$  divided by  $5 \times 10^3$   
 $= 1.356 \times 10^4$   
The product is in scientific notation.  
 $A = 1.356$  and  $N = 4$   
 $A + N = 1.356 + 4 = 5.356.$

3) 11 in base 2 + 222 in base 3 + 3333 in base 4 + 44444 in base 5  
Converting all to base 10:  
 $= [1(1) + 1(2)] + [2(1) + 2(3) + 2(9)] + [3(1) + 3(4) + 3(16) + 3(64)]$   
 $\quad + [4(1) + 4(5) + 4(25) + 4(125) + 4(625)]$   
 $= [1 + 2] + [2 + 6 + 18] + [3 + 12 + 48 + 192] + [4 + 20 + 100 + 500 + 2500]$   
 $= 3 + 26 + 255 + 3124$   
 $= 3408$  in base 10.  
Converting to base 6:  
 $3408 / 1296 = 2$  with remainder 816.  
 $816 / 216 = 3$  with remainder 168.  
 $168 / 36 = 4$  with remainder 24.  
 $24 / 6 = 4$  with remainder 0.  
Therefore, the base 6 equivalent is 23440.  
Check:  $2(1296) + 3(216) + 4(36) + 4(6) + 0(1)$   
 $= 2592 + 648 + 144 + 24 + 0$   
 $= 3408.$

Answers

- 1) 93  
2) 5.356  
3) 23440

**Category 4**  
**Arithmetic**  
**Meet #3 - January, 2023**



1) Find the value of this sum. Express your answer as a decimal.

$$2^{-2} + 2^{-1} + 2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

2) If  $\sqrt{25\%} - 3\% = X\%$  then what is the value of X ?

3) If  $\sqrt[3]{4^C} = 64$  then what is the value of C ?

**ANSWERS**

1) \_\_\_\_\_

2) \_\_\_\_\_

3) \_\_\_\_\_



**Solutions to Category 4**  
**Arithmetic**  
**Meet #3 - January, 2023**

$$\begin{aligned} 1) \quad & 2^{-2} + 2^{-1} + 2^0 + 2^1 + 2^2 + 2^3 + 2^4 \\ & = 0.25 + 0.5 + 1 + 2 + 4 + 8 + 16 \\ & = 31.75 \end{aligned}$$

$$\begin{aligned} 2) \quad & \sqrt{25\%} - 3\% = X\% \\ & \sqrt{0.25} - 0.03 = X\% \\ & 0.5 - 0.03 = X\% \\ & 0.47 = X\% \\ & X = 47 \end{aligned}$$

$$3) \quad \sqrt[3]{4^C} = 64$$

$$\begin{aligned} \text{Then } & 4^C = 64^3 \\ & 4^C = (4^3)^3 \\ & 4^C = 4^9 \\ & C = 9. \end{aligned}$$

**Answers**

1) 31.75

2) 47

3) 9

## Category 5

### Algebra

Meet #3 - January, 2023



- 1) If only positive integers can be used as values for  $N$  in the following inequality, then what is the sum of all possible values of  $N$  that make it true ?

$$3N - 5 < 29$$

- 2) How many integers make the following absolute value inequality true?

$$\left| \frac{23}{A+2} \right| \geq 4$$

- 3) The graph below is the set of all real values of  $W$  that make the following inequality true:  $|W - A| \leq C$



What is the value of  $3A + 5C$  ?

### Answers

1) \_\_\_\_\_

2) \_\_\_\_\_

3) \_\_\_\_\_

**Solutions to Category 5  
Arithmetic  
Meet #3 - January, 2023**

<u>Answers</u>	
1)	66
2)	10
3)	193

1)  $3N - 5 < 29$

$$3N < 34$$

$$N < 11.33333 \dots$$

The sum of the positive integers that satisfies this inequality is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11$$

$$= 66.$$

2)  $\left| \frac{23}{A+2} \right| \geq 4$

So,  $A + 2$  must fall between  $-5.75$  and  $5.75$  and  $A$  must fall between  $-7.75$  and  $3.75$ . The integers satisfying this condition, are  $-7, -6, -5, -4, -3, -1, 0, 1, 2,$  and  $3$ .

That is a total of **10** integers.

Note that the number  $-2$  was omitted from the list of solutions, as  $A = -2$  would make the denominator of the fraction equal to zero, thus making the fraction undefined.

3) First find the midpoint (average) of  $-13$  and  $45$  by computing half their difference and adding that number to  $-13$ :

$$(1/2)(45 - (-13)) = (1/2)(58) = 29.$$

$$\text{Then } -13 + 29 = 16.$$

So,  $16$  is the midpoint between  $-13$  and  $45$ .

Now, then inequality can be interpreted as, "The difference between any number,  $W$ , and the midpoint,  $A$ , is less than or equal to  $C$  units."

In this case,  $W$  is less than or equal to  $29$  units from the midpoint of  $16$ .

$$A = 16 \text{ and } C = 29.$$

$$\text{Then } 3A + 5C = 3(16) + 5(29)$$

$$= 48 + 145$$

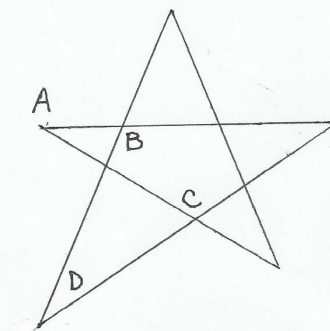
$$= 193.$$

**Category 6  
Team Round  
Meet #3 - January, 2023**

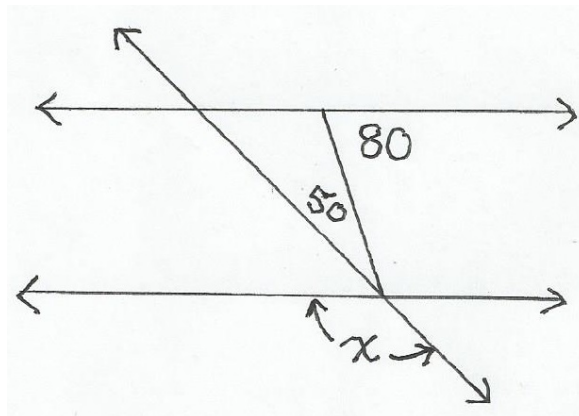
*Each of the following NINE problems is worth four points.*

- 1) A 500-foot-long train is about to enter a 500-foot-long tunnel. If the train is travelling at a rate of 500 feet per minute. then how many seconds will it take for the train to pass completely through the tunnel?
- 2) Three consecutive multiples of seven have a sum of 273. What is the value of the largest of the three multiples of seven?
- 3) What value of X makes the following equation true?  $X^X = 2^{2048}$

- 4) Three interior angles of this irregular pentagram are given as follows:  
 B = 112 degrees  
 C = 109 degrees  
 D = 37 degrees.  
 How many degrees are in the measure of exterior angle A if it is greater than 180 degrees?



- 5) The horizontal lines are parallel. The interior angles are 80 degrees and 50 degrees, as marked. How many degrees are in exterior angle X ?



- 6) If  $A - B = B - C = 2$ , then what is the value of  $\frac{(A-B)^2 + (B-C)^2}{(A-C)^2}$  ?

**ANSWERS**

1) \_\_\_\_\_

2) \_\_\_\_\_

3) \_\_\_\_\_

4) \_\_\_\_\_

5) \_\_\_\_\_

6) \_\_\_\_\_

7) \_\_\_\_\_

8) \_\_\_\_\_

9) \_\_\_\_\_

**SEE NEXT PAGE FOR PROBLEMS 7-9.**

7) If  $3x - 5y = 2$ , then what is the value of  $\frac{8^X}{32^Y}$  ?

8) The number 12345 is written in base 6. What is its value in base 9?

9) How many degrees are in the measure of one exterior angle of a regular 15-gon (a 15-sided polygon) ?

**Solutions to Category 6  
Team Round  
Meet #3 - January, 2023**

**ANSWERS**

- 1) 120
- 2) 98
- 3) 256
- 4) 320
- 5) 150
- 6) 0.5
- 7) 4
- 8) 2502
- 9) 24

1) It takes a full minute for the front of the train to reach the end of the tunnel and another minute for the back of the train to reach the end of the tunnel when the entire train is now out of the tunnel. Two minutes = 120 seconds.

2) Let the three consecutive multiples of 7 be represented by  $N$ ,  $N + 7$ , and  $N + 14$ . Then

$$N + (N + 7) + (N + 14) = 273$$

$$3N + 21 = 273$$

$$3N = 252$$

$$N = 84$$

and the largest is  $N + 14$ , or 98.

3)  $2^{2048} = (2^2)^{1024} = (2^4)^{512}$   
 $= (2^8)^{256} = 256^{256}$ . Therefore,  $X = 256$ .

4) The supplement of angle B is  $180 - 112$ , or 68 degrees which is a base angle of the triangle containing angle A. The supplement of angle C is  $180 - 109$ , or 71 degrees. That gives two angles of the triangle containing angle D. The remaining angle of that triangle is  $180 - (71 + 37)$ , or 72 degrees. Its vertical angle

is also 72 degrees, as vertical angles are congruent. That is the second numerical angle of the triangle containing angle A. Finding angle A of the triangle:  $180 - (68 + 72)$ , or 40 degrees. Finally, the exterior angle of A is  $360 - 40$ , or 320 degrees.

5) The interior angle on the same side of the transversal where the 80-degree angle is located is its supplement, or 100 degrees. The angle marked "X" is vertical to an angle that combines the 100-degree angle and the 50-degree angle, so equals the sum  $100 + 50$ , or 150 degrees.

6)  $A - B = 2$  and  $B - C = 2$ , so  $(A - B)^2 = 2^2 = 4$  and  $(B - C)^2 = 2^2 = 4$  as well. So, the numerator of the fraction has a value of  $4 + 4$ , or 8. For the denominator, Add  $A - B$  to  $B - C$  to get  $A - C$ . So,  $A - C = 4$ , and  $(A - C)^2 = 4^2 = 16$  and the decimal value of the fraction  $8/16$  is 0.5

$$7) \frac{8^X}{32^Y} = \frac{((2)^3)^X}{((2)^5)^Y} = \frac{2^{3X}}{2^{5Y}} = 2^{3X-5Y}$$

So, if  $3X - 5Y = 2$ , then  $2^{3X-5Y} = 2^2 = 4$ .

8) Converting 12345 base 6 to base 10  
 $= 5(1) + 4(6) + 3(36) + 2(216) + 1(1296)$   
 $= 5 + 24 + 108 + 432 + 1296$   
 $= 1865$  in base 10.

Converting to base 9:

The appropriate place values in base 9, from right to left, are 1, 9, 81, and 729.

$$1865 / 729 = 2 \text{ with remainder } 407.$$

$$407 / 81 = 5 \text{ with remainder } 2.$$

The nines place is 0.

Then skip ahead to the units place = 2.

So, the base 9 value is 2502.

9) The number of degrees in one exterior angle of a regular polygon is equal to 360 divided by the number of interior angles  
 $= 360 / 15 = 24$  degrees.