IMLEM Meet #4
February, 2020

Intermediate Mathematics League of Eastern Massachusetts

Calculator Meet

CLUSTER COORDINATORS - A reminder to all students of some of the rules and of appropriate behavior during this meet: • Many of you are guests in someone else's school – please be respectful of the classrooms and spaces you are using. Any "out of control" behavior in the halls or during a round is not acceptable. If an adult deems your behavior disrespectful or inappropriate, your score may not be counted. • CALCULATORS: only *scientific calculators* allowed for meets #4 & #5) • Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.

Category 1
Mystery
Meet #4 - February, 2020

Calculator Meet

1) The positive whole number ♡ has the following qualities:

- **is not an even number,**
- **is not divisible by 3,**
- **is not a prime number.**
- **is greater than 50.**
- **is not divisible by 11, and**
- **is less than 70.**

What is the value of ♡?

2) The five-digit number 6928N, where N is the units digit, is divisible by 9. What is the remainder when this five-digit number is divided by 4?

3) The sum of the squares of two consecutive odd integers is 3202. What is the value of the larger of those two consecutive odd integers?

Answers

1) _____

2) _____

3) _____

Solutions to Category 1
Mystery
Meet #4 - February, 2020

Answers	
1)	65
2)	2
3)	41

1) Eliminating the even numbers from the integers from 51 through 69 yields the following set of numbers:
51 53 55 57 59 61 63 65 67 69
Eliminating the primes gives this set:
51 55 57 63 65 69
Eliminating the multiples of three gives
55 65
Eliminating the lone multiple of 11 yields one number, namely 65.

Alternative solution:
Start by listing primes that can divide into the missing number and their product gets large very fast. 5,7,13 is as far as you get since 5*13 is too large. So the answer is either 35 or 65. Since you want the largest, the answer is 65. Easier!

2) The sum of the digits must be a multiple of 9 for the five-digit number to be divisible by 9. The sum of the first four digits is 6 + 9 + 2 + 8, or 25. The next two multiple of 9 are 27 and 36. Only a sum of 27 is possible, so N = 2 because N can't be 11. For 69282 to be divisible by 4, the number formed by the last two digits must be divisible by 4. The last two digits form the number 82 which, when divided by 4, gives a remainder of 2.

Alternative solution:
Divide (by hand) 6928N by 9. The last division in the problem is 9 into 7N. If 9 goes in evenly, the N=2. Divide 69282 by 4 and the remainder is 2.

SEE NEXT PAGE FOR SOLUTIONS TO #3.

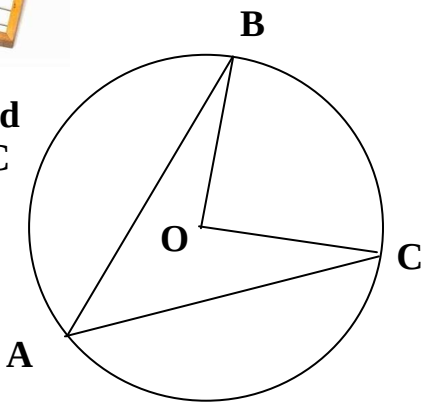
3) Using some number sense, students can get a good estimate of the two odd integers by squaring pairs of multiples of 10 and narrowing the choices. The sum of the squares of 20 and 20, for example, is 400 + 400, or 800 which is too small. The sum of the squares of 30 and 30 is 900 + 900, or 1800, also too small. The sum of the squares of 40 and 40 is 1600 + 1600, or 3200, which is very close. The sum of the squares of the odd integers surrounding 40, or the sum of the squares of 39 and 41, is 1521 + 1681, or 3202. Voila! The larger of the odd integers is 41.

Alternative solution:
For students who can solve quadratic equations, solve the quadratic N^2 +(N+2)^2 = 3202 which, using √1599 to help me, factored into (N-39)(N+42)=0. So N=39 and the larger odd integer is 41.

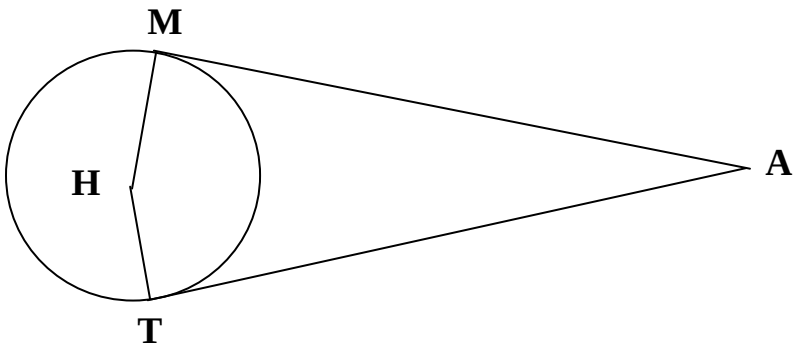
Category 2
Geometry
Meet #4 - February, 2020

Calculator Meet

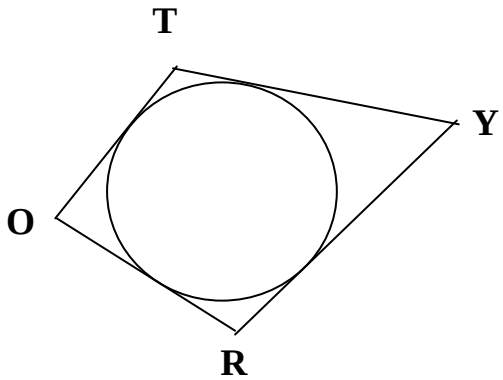
1) In a circle with center at point O, the inscribed angle, A, measures 37 degrees. Minor arc AC measures 114 degrees. How many degrees are in the measure of minor arc AB?



2) Segments HM and HT are radii of circle H. Segments AM and AT are tangent to circle H. The distance from A to H is 82 centimeters. The diameter of circle H is 36 centimeters. How many square centimeters are in the area of quadrilateral MATH ?



3) In the figure, four segments are tangent to the circle and form a quadrilateral. Segment TO = 15 inches. Segment OR = 18 inches. Segment RY = 22 inches. How many inches long is segment TY ?



Answers

1) _____ degrees

2) _____ sq cm

3) _____ inches

Solutions to Category 2
Geometry
Meet #4 - February, 2020

Answers	
1)	172
2)	1440
3)	19

1) Minor arc BC measures twice the angle measure of the inscribed angle, A, so 74 degrees. Minor arc AB is

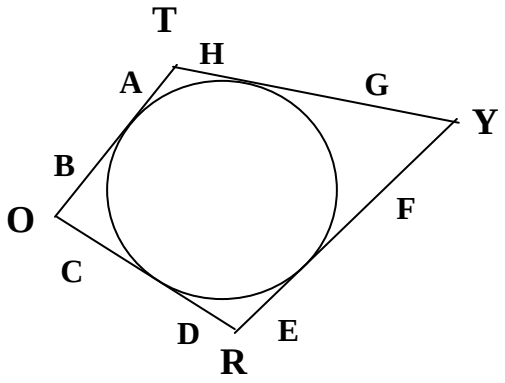
$$\begin{aligned} &360 - (BC + AC) \\ &= 360 - (74 + 114) \\ &= 360 - 188 \\ &= 172 \text{ degrees.} \end{aligned}$$

2) The radius of a circle is perpendicular to a tangent to that circle at the point of tangency. Therefore, radius HM is perpendicular to tangent MA and radius HT is perpendicular to tangent TA. Tangents to a circle from the same exterior point are congruent. So, MA = TA. Draw segment HA, creating two congruent triangles, HMA and HTA. If the diameter of the circle is 36 cm, then a radius is 18 cm. Each triangle is now twice a 9-40-41 right triangle, or an 18-80-82 right triangle. One could also use the Pythagorean Theorem to find the length of MA and TA, or 80 cm.
The area of quadrilateral MATH is twice the area of either triangle

$$\begin{aligned} &= 2\,(1/2)\,(\text{base})\,(\text{altitude}) \\ &= 2\,(1/2)\,(80)\,(18) \\ &= 1440 \text{ square centimeters.} \end{aligned}$$

3) My teacher colleague and test checker, Alice, likes to call this an "around-the-world" problem, as one strategy is to start at a one corner and go around the quadrilateral, labeling each segment until the length of the unknown segment is obvious. Also utilized is the property that tangents to a circle from the same exterior point are congruent. I will label the diagram with letters representing the segments of the tangents as A, B, C, D, E, F, G, and H as follows:

$$\begin{aligned} \text{Let} \quad & A = x \\ \text{Then} \quad & B = 15 - x \\ & C = 15 - x \\ & D = 18 - (15 - x) \\ & \quad = 3 + x \\ & E = 3 + x \\ & F = 22 - (3 + x) \\ & \quad = 19 - x \\ & H = A = x \end{aligned}$$



So, TY = H + G = x + (19 - x) = 19 inches.

Category 3
Number Theory ***Calculator Meet***
Meet #4 - February, 2020

1) The first six numbers in an arithmetic sequence are listed below:

7 10 13 16 19 22

If the pattern continues, then what is the value of the 43rd number in the sequence?

2) The first 21 digits of the decimal equivalent of the fraction $\frac{28}{91}$ are given in the following representation:

$= 0.307692307692307692307 \ldots$

What is the value of the 375th digit? (Note: The first digit is 3 and and the second digit is 0 and the third digit is 7, and so on.)

3) A formula for calculating the sum of the cubes of the first n positive integers is given below and to the right, where k = 1 is the first positive integer, or 1, and k = n is some integer larger than 1. For example, to find the sum of the cubes of the first four positive integers, one could simply do the grunt work and calculate $1^3+2^3+3^3+4^3=1+8+27+64=100$. Using the formula, instead, n = 4:

$$\frac{4^2(4+1)^2}{4}=\frac{16(25)}{4}=100.$$

What is the smallest value of n, such that the sum of the cubes of the integers 1 through n, inclusive, exceeds (or is greater than) 1,000,000?

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

Answers

1) _____

2) _____

3) _____

Solutions to Category 3
Number Theory
Meet #4 - February, 2020

Answers	
1)	133
2)	7
3)	45

1) The sequence is short enough that a student can list out to the 43rd term in short time. The nth term is 3n + 4, so the 43rd term is 3(43) + 4, or 133.

2) The decimal equivalent contains a repeating block of six digits. To find the 375th digit, divide 375 by 6. That gives a quotient of 62 with remainder 3. There are 62 complete blocks of the six digits, consuming 62(6), or 372 digits in all. The next three digits are 3, 0, and 7, with 7 being the 375th digit.

3) One strategy is to estimate a value of n that would get us close to 1,000,000. Since n and n+1 are relatively close in that neighborhood, the numerator's estimated value could be calculate as (n)(n)(n)(n), or the fourth power of n. The fourth powers of 10, 20, 30, 40, and 50 are 10,000, 160,000, 810,000, 2,560,000 and 6,250,000, respectively. When divided by four, each is now 2,500, 40,000, 202,500, 640,000, and 1,562,500. It is those final two that give us a reasonable starting point. Guessing and checking yields the smallest value of n, such that the sum of the cubes of whole numbers from 1 to n, inclusive, exceeds 1,000,000. Here are a few of the best calculations:

for the sum of cubes from . .

1 - 43 $\frac{43^2(44)^2}{4} = 894,916$

1 - 44 $\frac{44^2(45)^2}{4} = 980,100$

1 - 45 $\frac{45^2(46)^2}{4} = 1,071,225$

So, 45 is the smallest value of n that yields a sum greater than 1,000,000.

Category 4
Arithmetic
Meet #4 - February, 2020

Calculator Meet

1) At adulthood, Victoria had attained a height of 60 inches. When she was in the 6th grade, she was only 70% of what would become her adult height. How many inches tall was Victoria when she was in the 6th grade?

2) From the school year 2016-2017 to the school year 2017-2018, the Mansfield Qualters School experienced a 20% increase in the student population. The following school year, the student population decreased by 4% from the previous school year. If there were 250 students there in the school year 2016-2017, then how many were there in the school year 2018-2019?

3) The equation to the right calculates the value A, that is the result of investing an initial amount of money, M, at an annual (yearly) rate of R for T years when the interest is compounded W times annually. Robert would like to be a millionaire some day. How much money must Robert invest at an annual rate of 3.4% that compounds interest five times a month for 37 years? Round your answer up to the next hundred dollars to assure that Robert will have at least $1,000,000 after 37 years.

$$A = M\left(1+\frac{R}{W}\right)^{WT}$$

ANSWERS

1) _____ inches

2) _____ students

3) $ _____

Solutions to Category 4
Arithmetic
Meet #4 - February, 2020

Answers	
1)	42
2)	288
3)	284,400 or 284,400.00

1) 70% of 60 = (0.7)(60) = 42 inches.

2) 120% of 250 = (1.2)(250) = 300.
300 decreased by 4% = (300)(0.96) = 288.

3) $A = M\left(1+\frac{R}{W}\right)^{WT}$

$$1{,}000{,}000 = M\left(1+\frac{0.034}{(5)(12)}\right)^{(60)(37)}$$

$$1{,}000{,}000 = M\left(1+\frac{0.034}{(60)}\right)^{(2220)}$$

$$1{,}000{,}000 = M(1+0.0005666\ldots)^{(2220)}$$

$$1{,}000{,}000 = M(1.0005666\ldots)^{(2220)}$$

$$1{,}000{,}000 = M(3.5171238\ldots)$$

$$M = \frac{1{,}000{,}000}{3.5171238\ldots}$$

$$M \approx 284{,}323.23$$

$$M \approx 284{,}400$$

Therefore, Robert must invest $284,400, when rounded up to the next hundred dollars.

Category 5
Algebra ***Calculator Meet***
Meet #4 - February, 2020

1) At 2:00 A.M., Travis cast a shadow of 216 inches onto the ground while his cat's shadow was 48 inches long. The cat is 12 inches tall. How many inches tall is Travis?

2) A collection of nickels, dimes, and quarters has a total value of $ 22.10. The number of dimes is thirty less than twice the number of quarters. There are 29 more dimes than nickels. How many more quarters are there than nickels?

3) Lizzie and Simonne live 390 miles apart. They each leave their homes at the same time and head in each other's direction on the same road. Lizzie's average speed is 30 miles per hour less than that of Simonne. After three hours, they pass each other. Assume that each girl is travelling at her average rate of speed when they pass each other. How many miles per hour was Simonne travelling?

ANSWERS

1) _____ inches

2) _____

3) _____ mph

Solutions to Category 5
Algebra
Meet #4 - February, 2020

Answers	
1)	54
2)	8
3)	80

1) $\dfrac{\text{height}}{\text{length of shadow}} : \dfrac{12}{48} = \dfrac{T}{216}$

Cross products are equal whenever two ratios are equal:

$$48T = (12)(216)$$
$$48T = 2592$$
$$T = 54 \text{ inches}$$

2) Let X = **the number of quarters**
then $2X - 30$ = **the number of dimes**
and $2X - 30 - 29$ = **the number of nickels**

The total value of the coins is $ 22.10. Expressing this equation in terms of cents, rather than in dollars:

$$25X + 10(2X - 30) + 5(2X - 30 - 29) = 2210$$
$$25X + 20X - 300 + 10X - 150 - 145 = 2210$$
$$55X - 595 = 2210$$
$$55X = 2805$$
$$X = 51$$

So, there are 51 quarters, 72 dimes, and 43 nickels.
There are 8 more quarters than nickels.

3) (Rate of speed) x (Time) = (Distance)

	Rate	Time	Distance
Lizzie	$X - 30$	3	$3(X - 30)$
Simonne	X	3	$3X$

The total distance covered is 390 miles:

$$3X + 3(X - 30) = 390$$
$$3X + 3X - 90 = 390$$
$$6X - 90 = 390$$
$$6X = 480$$
$$X = 80$$

Simonne was travelling an average speed of 80 miles per hour.

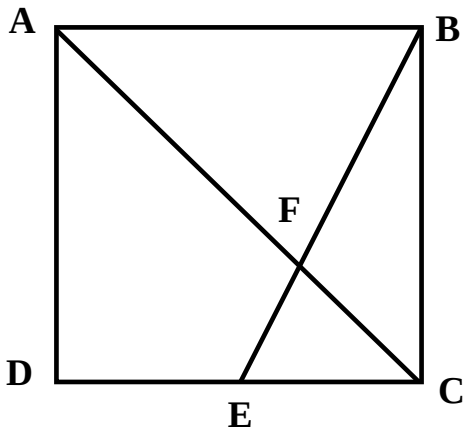
Category 6
Team Round
Meet #4 - February, 2020

Each of the following nine problems is worth <u>four</u> points.

1) Brenda tiled the floor of her recreation room with square-foot square tiles along the edges and square tiles measuring 2 feet by 2 feet for the rest of the floor. How many tiles did she use in all if the floor measures 16 feet by 24 feet?

2) The Sergeian mean of a set of numbers is the reciprocal of the sum of the reciprocals of the numbers in the set. What is the Sergeian mean of the numbers 1, 2, 4, and 8?

3) Point E is the midpoint of side CD in square ABCD. BE meets diagonal AC at F. The area of quadrilateral AFED is 45 square units. How many square units are in the area of square ABCD?



4) What is the value of the 214th term of this arithmetic sequence?
162 155 148 141 134 . . .

ANSWERS

1) _____

2) _____

3) _____

4) _____

5) _____

6) _____

7) _____

8) _____

9) _____

5) If it is now February, then what month is it 517 months from now?

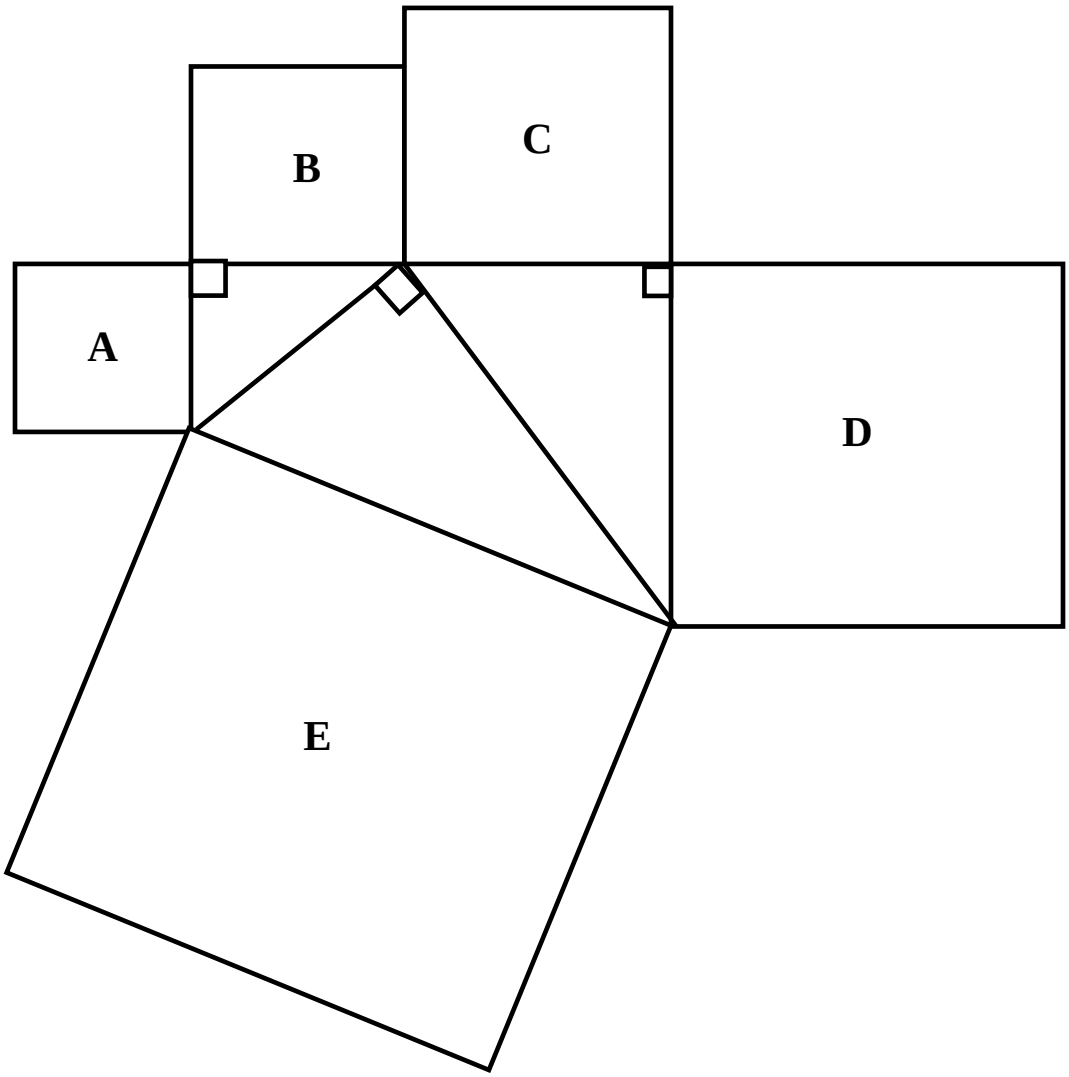
6) What is the sum of the whole numbers from 72 through 238, inclusive?

7) The sum of the squares of three consecutive multiples of three is 13,086. What is the value of the square root of the product of the three multiples of three, rounded to the nearest tenth?

8) If $2^N = 12$, then what is the value of 2^{N+6}?

PROBLEM #9 is on the NEXT PAGE.

9) The areas of squares A, B, C, and D are 17^2, 19^2, 23^2, and 29^2 square units, respectively. Each square has a side that is also the leg of a right triangle, as shown. How many square units are in the area of square E whose side is the hypotenuse of a right triangle, as shown? Your answer will be a whole number.



Solutions to Category 6
Team Round
Meet #4 - February, 2020

ANSWERS

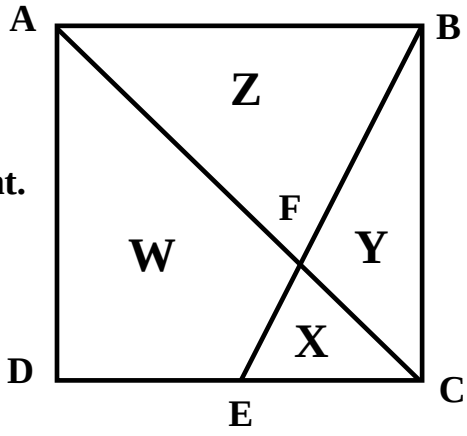
1) 153
2) $\frac{8}{15}$
3) 108
4) - 1329
5) March
6) 25,885
7) 535.6
8) 768
9) 2020

1) The perimeter of the room contains (2)(24) + (2)(16 - 2), or a total of 76 square-foot tiles. The interior of the floor contains (11)(7), or 77 tiles that each measure 2' by 2.' The total number of tiles is 76 + 77, or 153.

2) The reciprocal of the sum of the reciprocals of 1, 2, 4, and 8 is

$$\frac{1}{\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}} = \frac{1}{\frac{8}{8}+\frac{4}{8}+\frac{2}{8}+\frac{1}{8}} = \frac{1}{\frac{15}{8}} = \frac{8}{15}$$

3) Let the letters W, X, Y, and Z represent areas of polygons as shown to the right. Write sets of equations that represent facts about those spaces, solving for each letter,



accordingly. X + Y = one-quarter of square ABCD. X + W = one-half of square ABCD. Adding these two equations gives 2X + W + Y = 3/4 of square ABCD. Because X and Z are similar triangles, the ratio of their areas is equal to the square of the ratio of any pair of corresponding sides. Whereas the base of Z is twice that of the base of X, the area of Z is 4 times the area of X. So, we may replace Z with 4X. Square ABCD = W + X + Y + Z, or W + X + Y + 4X. Simplified, that is W + 5X + Y. Combine this equation with 2X + W + Y = 3/4 of square ABCD by subtracting, giving us 3X = 1/4 of square ABCD and 12X = the entire area of square ABCD. Now since X + W = 1/2 the area of square ABCD, then multiplying both sides by 2 yields 2X + 2W = the full area of square ABCD = 12X. Manipulating this last equation, 2X + 2W = 12X yields 2W = 10X

and 5X = W. Since W is the area of quadrilateral AFED = 45 (given), then 5X = 45 and A = 9. Then A + C = 9 + 45 = 54 = half the area of square ABCD. So, the full area of square ABCD = 108.

4) The value of the Nth term is 169 - 7N where the first term, 162, is 169 - 7(1), or 162. The value of the 214th term is 169 - 7(214), or 169 - 1498, or - 1329.

5) Every 12th month is also February. Divide 517 by 12 to get 43 with remainder 1. The 43 is the number of years and the 1 is the extra month after February, which is March.

6) Use the formula (N)(N + 1) / 2 to compute the sum of the whole numbers from 1 through 238, inclusive, and the sum of the numbers from 1 through 71, inclusive, and then subtract:

1 through 238: (238)(239)/2 = 28,441.
1 through 71: (71)(72)/2 = 2,556
Then 28,441 - 2,556 = 25,885.

7) Since the data generates a quadratic equation, then writing and solving that equation is effective. For students who prefer to "muck around" and use a combination of number sense and "guess and check," that can work even more quickly.
"Messing around" - take the square root of one-third of 13,086. That gives about 66. The squares of 63, 66, and 69, when added, yields 3969 + 4356 + 4761 = 13,086. Voila! The square root of the product of (63)(66)(69) is the square root of 286,902, or about 535.6323 . . . or, when rounded to the nearest tenth, yields 535.6.

8) Use properties of exponents and substitution:

$$2^{N+6} = \left(2^{N}\right)\left(2^{6}\right) = (12)(64) = 768.$$

9) The Pythagorean Theorem can be stated like this: The area of the square on the hypotenuse of a right triangle is equal to the sum of the areas on the legs. The area of the square adjacent to squares A and B 17^2+19^2. The area of the square adjacent to squares C and D is 23^2+29^2. Therefore, the area of square E is equal to the sum of the areas of the squares of that largest right triangle, or

$$17^2+19^2+23^2+29^2 = 289+361+529+841 = 2020.$$