

IMLEM Meet #4
February, 2019

Intermediate Mathematics League of Eastern Massachusetts



Calculator Meet

CLUSTER COORDINATORS - A reminder to all students of some of the rules and of appropriate behavior during this meet: • Many of you are guests in someone else's school – please be respectful of the classrooms and spaces you are using. Any “out of control” behavior in the halls or during a round is not acceptable. If an adult deems your behavior disrespectful or inappropriate, your score may not be counted. • **CALCULATORS:** only *scientific calculators* allowed for meets #4 & #5) • Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.

Category 1

Mystery

Meet #4 - February, 2019



Calculator Meet

- 1) When I was 12 years old, my sister was half my age. Now she is 37 years old. How many years old am I?

- 2) M is a positive integer and $7^2 + M^2 + 41^2 = 2019$. What is the value of M?

- 3) Simplify the expression below. Express your answer as a common fraction (reduced to lowest terms). Note: The three dots mean that you follow the pattern established by the first three factors and continue that pattern to the final three factors without skipping any factors. Note: The operation symbol between the 1 and the fraction is subtraction.

$$\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right) \dots \left(1 - \frac{1}{78}\right)\left(1 - \frac{1}{79}\right)\left(1 - \frac{1}{80}\right)$$

Answers

1) _____

2) _____

3) _____

**Solutions to Category 1
Mystery
Meet #4 - February, 2019**

1) My sister was 6 when I was 12. If she is now 37 years old, then I am 6 years older than her, or 43 years old.

$$\begin{aligned} 2) \quad 7^2 + M^2 + 41^2 &= 2019 \\ 49 + M^2 + 1681 &= 2019 \\ M^2 + 1730 &= 2019 \\ M^2 &= 289 \end{aligned}$$

Since M must be a positive integer, then M is the square root of 289, or 17.

<u>Answers</u>	
1)	43
2)	17
3)	$\frac{1}{40}$

$$\begin{aligned} 3) \quad &\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right) \cdots \left(1 - \frac{1}{78}\right)\left(1 - \frac{1}{79}\right)\left(1 - \frac{1}{80}\right) \\ &= \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right) \cdots \left(\frac{77}{78}\right)\left(\frac{78}{79}\right)\left(\frac{79}{80}\right) \end{aligned}$$

Simplify, or "reduce", or "cancel" diagonal like factors, as follows . . .

$$= \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right) \cdots \left(\frac{77}{78}\right)\left(\frac{78}{79}\right)\left(\frac{79}{80}\right)$$

. . . thus leaving the expression

$$\frac{2}{80} \quad \text{that simplifies to the common fraction } \frac{1}{40}.$$

Category 2

Geometry

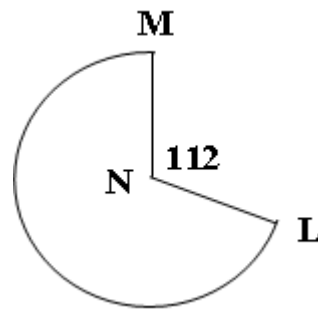
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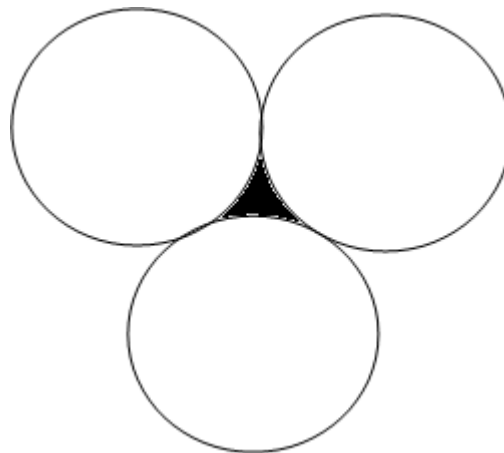
Calculator Meet

1) The diameter of a circle measures 16 centimeters. Its circumference is A centimeters and its area is B square centimeters. What is the value of B - A ? Use $\pi \approx 3.14$.

2) In the partial circle to the right, point N is the center. If NM and NL are radii and each radius measures 6.7 feet. How many square feet are in the area of this figure? Use $\pi \approx 3.1416$. Round your final answer to the nearest tenth of a square foot. The central angle measures 112 degrees.



3) Three congruent (identical) circles are tangent to one another, as shown. (Two circles are tangent if they intersect at only one point.) The radius of each circle is 26 inches. How many square inches are in the area of the black region? Use $\pi \approx 3.1$. Round your final answer to the nearest whole number of square inches.



Answers

1) _____

2) _____ sq. ft.

3) _____ sq. in.

Solutions to Category 2
Geometry
Meet #4 - February, 2019

<u>Answers</u>	
1)	150.72
2)	97.2
3)	123

1) $A = \text{circumference} = (\pi) (\text{diameter})$
 $= (3.14) (16)$
 $= 50.24 \text{ cm.}$

$B = \text{area} = (3.14) (8) (8)$
 $= 200.96 \text{ sq. cm.}$

$B - A = 200.96 - 50.24 = 150.72.$

2) $\text{Area of figure} = (\text{fraction of circle}) (\text{area of circle})$
 $= (248 / 360) (3.1416) (6.7) (6.7)$
 $= 97.15152 \dots$
 $= 97.2 \text{ square feet when rounded to the nearest tenth.}$

3) Draw radii through points of tangency to create an equilateral triangle, each of whose central angles measures 60 degrees. Find the area of the triangle, then subtract the areas of the three sectors to get the area of the black region.

$\text{area of triangle} - \text{area of three sectors}$
 $= (\text{side squared divided by } 4) (\text{square root of } 3) - (3) (1/6) (3.1) (26) (26)$
 $= [(52) (52) / 4] (1.73205 \dots) - 1047.8$
 $= (1170.865 \dots) - 1047.8$
 $= 123.06 \dots$
 $= 123 \text{ when rounded to the nearest whole number of square inches.}$

Category 3

Number Theory

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Calculator Meet

- 1) What is the value of the 30th term (number) in the following arithmetic sequence?

17 20 23 26 29 . . .

- 2) Kenton's last birthday was 236 days ago. If today is Thursday, then on what day of the week was Kenton's last birthday?

- 3) The series below is the sum of consecutive integers whose first term is 26 and whose final term is 672 and contains all of the positive integers between 26 and 672 as well. What is that sum?

$$26 + 27 + 28 + 29 + . . . + 671 + 672$$

Answers

1) _____

2) _____

3) _____

**Solutions to Category 3
Number Theory
Meet #4 - February, 2019**

<u>Answers</u>	
1)	104
2)	Saturday
3)	225,803

1) Since there is a difference of 3 between any two consecutive terms, there is a connection between this sequence and the sequence of multiples of 3, following this formula: $3N + 14$, where N is the number of the term. The 30th term is $3(30) + 14$, or 104.

2) Divide 236 by 7 (the number of days in a week) to get 33 with remainder 5. Kenton was born 33 weeks ago . . . plus another 5 days in the past. Counting backwards 5 days from Thursday yields an answer of Saturday.

3) The sum of the given numbers can be considered as the difference between the sum of the integers from 1 through 672, inclusive, and the sum of the integers from 1 through 25, inclusive.

$$\begin{aligned} & (1 + 2 + 3 + \dots + 672) - (1 + 2 + 3 + \dots + 25) \\ &= [(n)(n + 1) / 2] - [(m)(m + 1) / 2] \\ &= [(672)(673) / 2] - [(25)(26) / 2] \\ &= 226,128 - 325 \\ &= 225,803 \end{aligned}$$

A common strategy is to add opposite ends of the series, working your way toward the middle:

$$\begin{aligned} & (26 + 672) + (27 + 671) + (28 + 670) + \dots \\ &= 698 + 698 + 698 + \dots \end{aligned}$$

There are as many sums of 698 as half the number of terms in the series. That number of sums is $(672 - 26 + 1) / 2$, or $647 / 2$, or 323.5. Then $(323.5)(698) = 225,803$, the same result as by using the formula above.

Category 4

Arithmetic

Meet #4 - February, 2019



Calculator Meet

1) Carolyn treated her friends to lunch. The bill came to \$ 28.60. There was a 5% meals tax added to the bill. Carolyn also left a 20% tip (based on the \$ 28.60 cost of the meal). How much money did this lunch cost Carolyn in all, including meals, tax, and tip?

2) Pele was injured in a soccer game. During many months of recovery, he lost 20% of his body weight. Once he began to train again to get into shape to play soccer again, he regained all the weight that he had lost. By what percent did his weight increase during the time Pele was training to get back into shape?

3) The equation to the right calculates the value A , that is the result of investing an initial amount of money, M , at an annual (yearly) interest rate of R for T years when the interest is compounded W times annually. Alexandria wanted to invest \$80,000 at her bank at an annual interest rate of 3.7%, compounded annually (yearly), for 12 years. Kamala insisted that she would earn more interest at her bank at an annual rate of 3.2%, compounded three times monthly, for 12 years. Who was correct? Also, how much more interest was earned with the more profitable deal? Round your answer to the nearest hundred dollars. Note: *You must get both parts correct to earn credit for this problem.*

$$A = M \left(1 + \frac{R}{W} \right)^{WT}$$

ANSWERS

1) \$ _____

2) _____ %

3) who? _____

\$ _____

Solutions to Category 4
Arithmetic
Meet #4 - February, 2019

<u>Answers</u>	
1)	35.75
2)	25 (%)
3)	Alexandria AND 6,300

- 1) The total of the meals, tax, and tip is equal to 125% of 28.60, or $(1.25)(28.6) = 35.75$.
- 2) For a student who has not yet studied algebra, an effective strategy is to start with a number like 100 pounds. Losing 20% brings Pele's weight down to 80 pounds. Then 25% of 80 added to 80 yields $20 + 80$, or 100 pounds.

3)

$$A = M \left(1 + \frac{R}{W} \right)^{WT}$$

$$A = 80,000 \left(1 + \frac{0.032}{36} \right)^{(36)(12)}$$

$$A = 80,000(1.0008888\dots)^{432}$$

$$A = 80,000(1.46789504\dots)$$

$$A = 117,431.6038$$

This is how much money Kamala would have after 12 years.

Now for Alexandria:

$$A = M(1+R)^T \quad \text{This is the simpler version, as } W = 1.$$

$$A = 80,000(1+0.037)^{12}$$

$$A = 80,000(1.037)^{12}$$

$$A = 80,000(1.546482738\dots)$$

$$A = 123,718.619$$

This is how much money Alexandria would have after 12 years.

The difference would be $123,718.619 - 117,431.6038$, or \$6,287.0152, with Alexandria having earned more. Rounding the final answer to the nearest hundred yields \$6,300. To earn two points credit, students must have both answers correct - Alexandria AND 6,300.

Category 5

Algebra

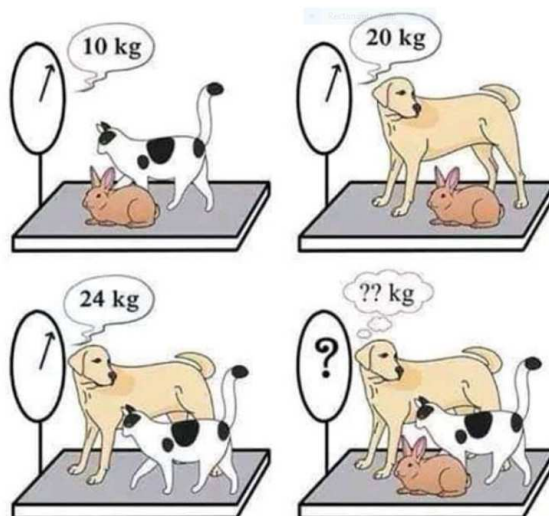
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Calculator Meet

- 1) There are three consecutive positive integers, such that the largest is 44 less than twice the smallest. What is the value of the smallest of these three consecutive integers?

- 2) Michael ran 10 miles in 80 minutes. Brady ran 16 miles in 80 minutes. How many more miles did Brady run than Michael in 2.5 hours if they each maintained their same rate of speed? Express your answer as a decimal.

- 3) The picture below was discovered in the Internet. Together, the cat and the rabbits weigh a total of 10 kg (kilograms), the dog and rabbit weigh a total of 20 kg, and the dog and cat weigh a total of 24 kg. In all, how much do the dog, cat, and rabbit weigh?



ANSWERS

1) _____

2) _____ miles

3) _____ kg

Solutions to Category 5

Algebra

Meet #4 - February, 2019

<u>Answers</u>	
1)	46
2)	11.25
3)	27

- 1) Let X , $X + 1$, and $X + 2$ represent three consecutive positive integers.

$$X + 2 = 2X - 44$$

$$46 = X$$

So, the smallest integer is 46.

- 2) Michael's rate is 10 miles per 80 minutes, or 1 mile per 8 minutes. In 2.5 hours, or 150 minutes, he can run $150 / 8$, or 18.75 miles. Brady's rate is 16 miles per 80 minutes, or 1 mile per 5 minutes. In 150 minutes, Brady can run $150 / 5$, or 30 miles. So, in 2.5 hours, Brady can run $30 - 18.75$, or 11.25 miles farther.

- 3) Although the answer to this problem can be deciphered without writing formal equations, the following algebraic solution involves solving a system of equations in three variables.

Let D = the weight of the dog in kg

C = the weight of the cat in kg

R = the weight of the rabbit in kg

Then $C + R = 10$

$$D + R = 20$$

$$D + C = 24$$

Adding the three equations produces $2C + 2D + 2R = 54$

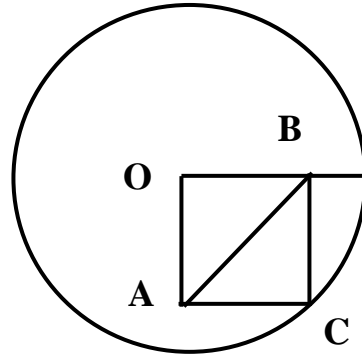
Dividing both sides by 2 yields $C + D + R = 27$

Therefore, the three animals weigh a total of 27 kg.

**Category 6
Team Round
Meet #4 - February, 2019**

Each of the following nine problems is worth four points.

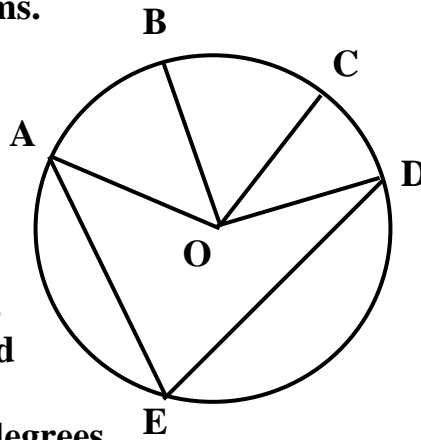
1) In the circle to the right, point O is the center, OBCA is a rectangle and AB measures 11.7 inches. How many inches are in the circumference of the circle? Use $\pi = 3.14$.



2) The 12-digit whole number 346,904,352,81W is divisible by 12. What is the value of the digit W if it is the units digit of the 12-digit number?

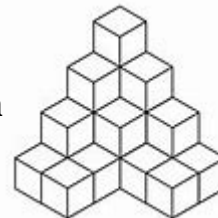
3) The average of ten positive numbers is 240. Each of the ten numbers is increased by 12.5%. What is the average of the new set of ten numbers?

4) There are three rational numbers, 0.5, $\frac{1}{3}$, and C, such that their average is the number one. What is the value of C? Express your answer as an improper fraction in lowest terms.



5) In the circle at the right, angle AOB measures 48 degrees, angle COD measures 37 degrees, and angle AED measures 72 degrees. How many degrees are in central angle BOC?

6) How many small cubes are in this figure? Each column of cubes sits on the floor, or rests on the same plane.



7) If $11^2 + 23^2 + C^2 = 2019$, then what is the value of C if it is a negative integer?

ANSWERS

1) _____

2) _____

3) _____

4) _____

5) _____

6) _____

7) _____

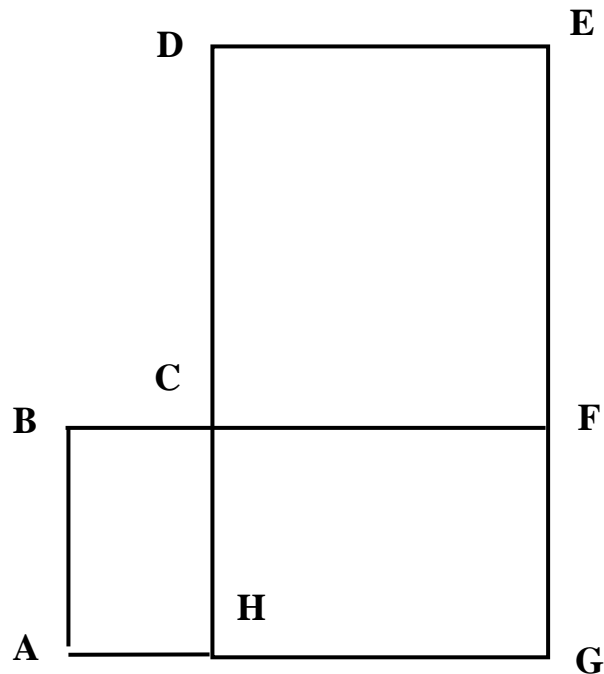
8) _____

9) _____

- 8) In a geometric sequence, the ratio of any two consecutive terms is the same as the ratio of any other two consecutive terms, in the same order. In the following geometric sequence, A, B, and C are three (not the first three) of the first six terms. What is the value of $A + B + C$?

7 A B 448 C 7168 . . .

- 9) The diagram below represents the neighborhood where letter carrier, Mr. McFeeley, delivers mail. Each letter corresponds to a street corner. On Monday, his route is ABCFGHA and is 14 km (kilometers) long. On Tuesday, his route is HCDEFGH and is 22 km long. On Wednesday, his route is ABCDEFGHA and is 24 km long. On Thursday, he has a much shorter route: CFGHC. How many km long is Mr. McFeeley's Thursday route? Notes: All corners are right angles. $FG = GH$. The figure is not drawn to scale.



**Solutions to Category 6
Team Round
Meet #4 - February, 2019**

ANSWERS

1) 73.476

2) 6

3) 270

4) $\frac{13}{6}$

5) 59

6) 22

7) -37

8) 1932

9) 12

1) The diagonals of a rectangle are congruent. So, $AB = OC =$ the radius of the circle = 11.7 in.
The circumference = $2(3.14)(11.7) = 73.476$ in.

2) To be divisible by 12, this number must be divisible by 4 and by 3. The final two digits, 1W, must be divisible by 4, so W could be 2 or 6. The sum of the 12 digits must be divisible by 3. Only when $W = 6$ does that happen, as that sum is 51. So, $W = 6$.

3) 12.5% is equivalent to the unit fraction $1/8$.
If each of the ten numbers is increased by $1/8$, then so is the sum. $240 + (1/8 \text{ of } 240)$ is 270.

4) $(0.5 + 1/3 + C) / 3 = 1$
 $0.5 + 1/3 + C = 3$
 $C = 3 - (0.5 + 1/3)$
 $C = 3 - (5/6)$
 $C = 13/6$

5) If inscribed angle $AED = 72$ degrees, then arc $ABCD =$ twice that, or 144 degrees, making central angle $AOD = 144$ degrees. The three angles that comprise angle AOD , therefore, have a sum of 144 degrees.

Angle $AOB +$ angle $BOC +$ angle $COD = 144$

$48 +$ angle $BOC + 37 = 144$

angle $BOC + 85 = 144$

angle $BOC = 59$

Therefore, central angle BOC measures 59 degrees.

6) The bottom tier contains 12 cubes. The tier above it contains 6 cubes. The one above that contains 3 cubes and the top is just 1 cube. The total number of cubes is $12 + 6 + 3 + 1 = 22$ cubes.

THE SOLUTIONS TO PROBLEMS #7, 8, AND 9 - NEXT PAGE

$$\begin{aligned}
7) \quad & 11^2 + 23^2 + C^2 = 2019 \\
& 121 + 529 + C^2 = 2019 \\
& 650 + C^2 = 2019 \\
& C^2 = 1369 \\
& C = \pm\sqrt{1369} \\
& C = \pm 37
\end{aligned}$$

But since the question asks for C to be a negative integer, then the only correct answer is -37 .

8) It requires two factors of the common ratio to jump from 448 to 7168 yet three jumps to get from 7 to 448. It is easier to deal with the first scenario: $448 (X) (X) = 7168 \dots (X) (X) = 16 \dots$ so $X = 4$, the common ratio. Then $7 (4) = A = 28$ and $A (4) = 28 (4) = B = 112$. Also, $448 (4) = C$, so $C = 1792$. Checking: $7 (4) (4) (4) = 448$. Yay! Finally, $A + B + C = 28 + 112 + 1792 = 1932$.

9) For convenience, side lengths are marked with the letters x , y , and z , with congruencies as noted.

$$\begin{aligned}
4x + 2y &= 14 \\
4x + 2z &= 22 \\
4x + 2y + 2z &= 24
\end{aligned}$$

Combining the first and third equations, via subtraction, yields $2z = 10$, so $z = 5$.

$$\begin{aligned}
4x + 10 &= 22, \text{ so } 4x = 12 \\
\text{and } x &= 3.
\end{aligned}$$

Therefore, route CFGHC is $4(3)$, or 12 km.

