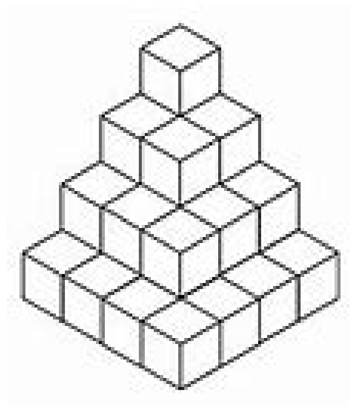
IMLEM Meet #3
January, 2019

Intermediate Mathematics League of Eastern Massachusetts

CLUSTER COORDINATORS - A reminder to all students of some of the rules and of appropriate behavior during this meet: • Many of you are guests in someone else's school – please be respectful of the classrooms and spaces you are using. Any "out of control" behavior in the halls or during a round is not acceptable. If an adult deems your behavior disrespectful or inappropriate, your score may not be counted. • No calculators (or only scientific calculators allowed for meets #4, #5) • Everyone take a moment to turn off any electronic devices that you want to have with you during the rounds. No electronic devices may be on during the rounds. Use of these devices during the rounds will result in a disqualification.

Category 1
Mystery
Meet #3 - January, 2019

1) How many of the smallest cubes are in this figure? Each column of cubes sits on the floor, or rests on the same plane.



2) Twelve matching pairs of socks, with each pair different from each of the remaining eleven pairs, are thrown into the washing machine. After washing and drying the socks, Oliver randomly reaches into the dryer and selects socks one at a time in hopes of selecting a matching pair. What is the least number of socks that Oliver must remove from the dryer in order to guarantee that he has at least one matching pair of socks?

3) The square of an integer is the result of multiplying the integer by itself. For example, the square of 7 is 49, because 7x7 = 49. The sum of the squares of three consecutive positive odd integers is 683. What is the largest of those three consecutive positive odd integers?

Answers

1) ______

2) ______

3) ______

Solutions to Category 1
Mystery
Meet #3 - January, 2019

Answers
1) 30
2) 13
3) 17

1) Top layer: 1
Second layer: 4
Third layer: 9
Fourth layer: 16
Total: 1 + 4 + 9 + 16 = 30

2) Worst case scenario for Oliver: that each sock he withdraws from the dryer is different from all those previously withdrawn. The maximum of such individual socks withdrawn is twelve. The next sock drawn must match one previously drawn. Hence, the answer is 13 socks.

3) Students can guess and check their way to this answer within the time allotted. Those who apply an algebraic strategy may do the following:
Let X = the smallest of three consecutive positive odd integers
Then X + 2 = the next larger positive odd integer
and X + 4 = the largest of the three consecutive positive odd integers.

$$X^2+(X+2)^2+(X+4)^2=683$$
$$X^2+X^2+4X+4+X^2+8X+16=683$$
$$3X^2+12X+20=683$$
$$3X^2+12X-663=0$$
$$X^2+4X-221=0$$
$$(X+17)(X-13)=0$$
$$X=13 \text{ or } X=-17$$

Only 13 satisfies the condition that the integer is both positive AND odd. Therefore, the three consecutive positive odd integers are 13, 15, and 17, with the largest being 17. The answer is 17.

Note: Even for students adept at algebra, the algebraic approach requires knowledge of solving quadratic equations. The "guess, check, and revise" technique, along with good number sense, should produce a quicker result.

Category 2
Geometry
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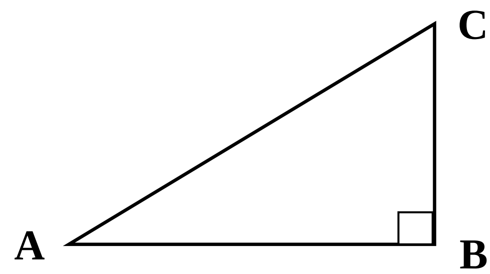
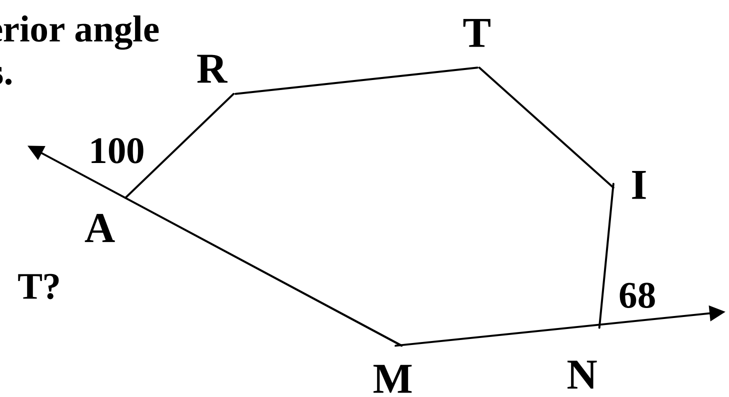
1) In right triangle ABC, leg AB = 8 units and leg BC = 6 units. How many units long is the hypotenuse, AC?



2) Hexagon MARTIN has exterior angle measures as given, in degrees. Interior angles R, T, M, and I are congruent (and have the same measure). How many degrees are in the measure of interior angle T?



3) Jackson lives in Northborough. He walked 14 miles to the east, then 3 miles south, 22 miles east, and finally 12 miles south. Jackson walked an average speed of 3 miles per hour. If, instead, he had walked directly in a straight line from his starting point to his final destination, then how many hours fewer would it have taken him to do his walk?

Answers

1) _______ units

2) _______ degrees

3) _______ hours

Solutions to Category 2
Geometry
Meet #3 - January, 2019

Answers
1) 10
2) 132
3) 4

1) Using the Pythagorean Theorem:

$$(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$$
$$(6)^2 + (8)^2 = (h)^2$$
$$36 + 64 = (h)^2$$
$$100 = (h)^2$$
$$10 = h$$

Therefore, AC = 10 units.

2) The sum of the interior angles of a convex polygon is 180 (n - 2), where n is the number of sides. Calculating the total number of degrees of the interior angles: 180 (6 - 2) = 180 (4) = 720 degrees. The given exterior angles of 100 and 68 degrees, respectively, have supplementary angles that are interior to the hexagon, and therefore measure 80 and 112 degrees, respectively. The total of the remaining four congruent interior angles, namely R, T, M, and I, is 720 - (80 + 112), or 720 - 192, or 528 degrees. Dividing 528 by 4 yields 132 degrees, the measure of each of the remaining interior angles. Therefore, angle T measures 132 degrees.

3) Adding the east distances yields 14 + 22 = 36 miles. Adding the south distances yields 3 + 12 = 15 miles. The distance from starting point to final destination is the hypotenuse of a right triangle with legs 36 and 15 miles. Use the Pythagorean Theorem to find the length of the hypotenuse:

$$15^2 + 36^2 = d^2$$
$$225 + 1296 = d^2$$
$$1521 = d^2$$
$$39 = d$$

Oliver originally walked 15 + 36, or 51 miles. By taking the hypotenuse shortcut, he walked 12 miles fewer (51 - 39 = 12). At 3 miles per hour, he walked 4 hours fewer (12 / 3 = 4).

Category 3
Number Theory
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1) Write the base 2 numeral 11011 as a base 10 numeral.

2) There are approximately 100 trillion, that is 1×10^{14}, cells in a typical (average) human body. There are about as many atoms in a typical cell. Using these figures, about how many atoms are in a typical human body? If that number is written in scientific notation as $A \times 10^{B}$, then what is the value of A + B? Express your answer as a whole number.

3) Write the base 7 numeral 2436 as a base 3 numeral.

Answers
1) ________
2) ________
3) ________

Solutions to Category 3
Number Theory
Meet #3 - January, 2019

Answers	
1)	27
2)	29
3)	1020200

1) The base 2 numeral 11011 has a base 10 value, from right to left:

1(1) + 1(2) + 0(4) + 1(8) + 1(16)
= 1 + 2 + 0 + 8 + 16
= 27

2) Multiply the number of atoms in a cell by the number of cells in a human body:

$$(1 \times 10^{14}) \times (1 \times 10^{14})$$
$$= 1 \times 10^{28}$$

So, A = 1 and B = 28, so A + B = 1 + 28 = 29.

3) Converting the base 7 numeral to a base 10 numeral, from right to left:

6(1) + 3(7) + 4(49) + 2(343)
= 6 + 21 + 196 + 686
= 909

Converting 909 base 10 to base 3:
First, divide 909 by the highest power of 3 that is less than 909, namely, 729. Then subtract 729 from 909, yielding 180. Now divide 180 by the highest power of 3 that is less than 180, namely 81, which is a power of 3. We get two 81s for a total of 162. Subtract 162 from 180 to get 18. With that, we get two 9s. Now, fill in zeroes for the powers of 3 for which we have none.
So we have 1(729) + 0(243) + 2(81) + 0(27) + 2(9) + 0(3) + 0(1), or 1020200.
Checking: 729 + 0 + 162 + 0 + 18 + 0 + 0 = 909. Check!

Category 4
Arithmetic
Meet #3 - January, 2019

1) If $3^N = 729$, then what is the value of N?

2) Evaluate: $\sqrt{36} + 4\sqrt[3]{125} - 3\sqrt[4]{16} + 7\sqrt[5]{100{,}000}$

3) What is the value of

$$\sqrt[3]{8\sqrt{121} + 2\sqrt{\sqrt[3]{1331} + \sqrt[6]{512}} + \sqrt{625}}\ ?$$

ANSWERS

1) ________

2) ________

3) ________

Solutions to Category 4
Arithmetic
Meet #3 - January, 2019

Answers	
1)	6
2)	90
3)	5

1) 3 x 3 x 3 x 3 x 3 x 3 = 729, so N = 6.

2) $\sqrt{36} + 4\sqrt[3]{125} - 3\sqrt[4]{16} + 7\sqrt[5]{100{,}000}$

$= 6 + 4(5) - 3(2) + 7(10)$
$= 6 + 20 - 6 + 70$
$= 90$

3) $\sqrt[3]{8\sqrt{121} + 2\sqrt{\sqrt[3]{1331} + \sqrt[6]{5^{12}}} + \sqrt{625}}$

$= \sqrt[3]{8(11) + 2\sqrt{11 + 25} + 25}$

$= \sqrt[3]{88 + 2\sqrt{36} + 25}$

$= \sqrt[3]{88 + 2(6) + 25}$

$= \sqrt[3]{88 + 12 + 25}$

$= \sqrt[3]{125}$

$= 5$

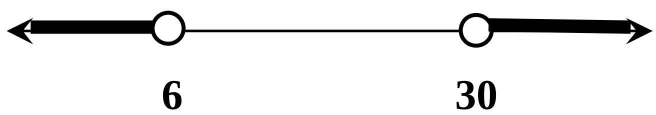
Category 5
Algebra
Meet #3 - January, 2019

1) Find the value of this absolute value expression:

$$|7| + |-10| + |0|$$

2) The graph below represents all values of X that make the accompanying absolute value inequality true. What is the value of A?

$$|X - A| > 12$$



3) For the inequality - 7 - 3 (2N - 5) < 38, the domain of N is {all negative integers}. In other words, only negative integers can be considered as replacements for N to make the inequality true. What is the sum of all possible values of N that make this inequality true?

ANSWERS

1) _______

2) _______

3) _______

Solutions to Category 5
Algebra
Meet #3 - January, 2019

Answers

1) 17

2) 18

3) - 10

1) $|7| + |-10| + |0|$

$= 7 + 10 + 0$
$= 17$

2) $|X - A| > 12$

This absolute value inequality can be interpreted as follows: "The distance between a value of X and A is more than 12 units." A must be the midpoint of 6 and 30, namely, 18. So, the distance between 18 and 6 is 12 units and the distance between 18 and 30 is also 12 units. All values to the left of 6 or to the right of 30 are greater than 12 units from 18. Therefore, A is 18.

3) $-7 - 3(2N - 5) < 38$

$-7 - 6N + 15 < 38$	Distribute the 3
$8 - 6N < 38$	Combine like terms
$-6N < 30$	Subtracting 8 from both sides
$N > -5$	Dividing both sides by - 6 reverses the sense of the inequality (< becomes >).

Since the domain is {negative integers}, the solution includes the numbers - 4, - 3, -2, and - 1. The sum of these numbers is - 10.

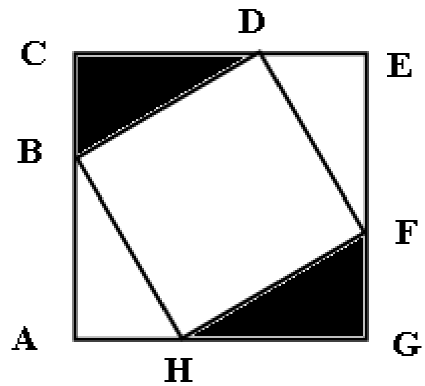
Category 6
Team Round
Meet #3 - January, 2019

Each of the following nine problems is worth four points.

1) 5! means (5) (4) (3) (2) (1). In general, N! means (N) (N - 1) (N - 2) . . . (3) (2) (1). What is the value of the expression to the right?

$$\frac{(7!)(8!)}{(5!)(6!)}$$

2) Square ACEG has a perimeter of 148 inches. Square BDFH has a perimeter of 124 inches. How many square inches, in total, are in the shaded area?



3) Erin is E years old in the year E^2. If the 4-digit year E^2 is in the 20th century, then in what year was Erin born?

4) How many prime numbers less than 100 have a units digit (ones place) of 7?

5) How many whole numbers are between $\sqrt{58}$ and $\sqrt{207}$?

ANSWERS

1) ________

2) ________

3) ________

4) ________

5) ________

6) ________

7) ________

8) ________

9) ________

6) $X^Y + Y^X = 1000$. What is the positive difference between X and Y? Both X and Y are positive integers.

7) The sum of the squares of two positive integers is 3341. Their difference is 891. What is the sum of the two original integers?

8) What 2-digit whole number has a value that is three times the sum of its digits?

9) Arrange the digits 0, 1, 2, and 3 to form a 4-digit number that is divisible by the smallest four prime numbers.

Solutions to Category 6
Team Round
Meet #3 - January, 2019

ANSWERS

1)	2352
2)	204
3)	1892
4)	6
5)	7
6)	998
7)	81
8)	27
9)	2310

1) $\frac{(\ )(\ )}{(\ )(\ )} = \frac{(7)(6)(5!)(8)(7)(6!)}{(5!)(6!)}$

= (7)(6)(8)(7) = 2352

2) Divide the perimeter of each square by 4 to give the length of one side. For the large square, 148 / 4 = 37 inches. For the smaller square, 124 / 4 = 31 inches. The area of the large square is (37)(37), or 1,369 square inches. The area of the smaller square is (31)(31), or 961 square inches. The difference in the two areas is 1,369 - 961, or 408 square inches. This represents the area of the four triangles surrounding the small square. Divide 408 by 2 to get the shaded area, or 204 square inches. (Note: The four triangles are congruent via definition of square, complementary and supplementary angles, and AAS.)

3) Guessing and checking reveals that the only square number in the 20th century is (44)(44), or 1936. If Erin is 44 years old in 1936, then she was born in the year 1936 - 44, or 1892.

4) These numbers qualify: 7, 17, 37, 47, 67, 97. Therefore, there are six prime numbers less than 100.

5) These numbers qualify: 8, 9, 10, 11, 12, 13, and 14. Therefore, there are seven such numbers.

6) The only numbers that qualify are 1 and 999. X - Y = 999 - 1 = 998. I am unaware of other solutions.

THE SOLUTIONS TO #7, #8, AND #9 ARE ON THE NEXT PAGE.

7) If W and P are the two original integers, then this system of two equations can be utilized to produce a solution: $W^2 + P^2 = 3341$ and $W^2 - P^2 = 891$. Adding the two equations yields $2W^2 = 4232$. Then $W^2 = 2116$ and W = 46 and P = 35. The sum of the two original integers is 46 + 35, or 81.

8) Let U = the units digit and T = the tens digit.

U + 10T = 3 (U + T)
U + 10T = 3U + 3T
7T = 2U

If T and U must be whole numbers - as they are digits - then T = 2 and U = 7 and the two-digit number is 27.

9) The four smallest prime numbers are 2, 3, 5, and 7. Among the four given digits 0, 1, 2, and 3, the 0 must be the units digit so that the four-digit number is divisible by both 5 and 2. The sum of the digits is 6, so that any arrangement of the remaining digits would result in a number divisible by 3. Guessing and checking yields the number 2310 that is also divisible by 7.