

IMLEM Meet #4  
March, 2018

# Intermediate Mathematics League of Eastern Massachusetts





## Solutions to Category 1

### Mystery

#### Meet #4 - March, 2018

1) Since person #11 is opposite person #1, then rotating clockwise three people puts person #14 opposite person #4.

2) Using the given four digits, the final 4-digit number must have a units digit of 0 in order to be divisible by 2 and by 5. That leaves only six possibilities for arranging the remaining three digits. Since the sum of the digits is 6, the 4-digit number is divisible by 3, regardless of the placement of the digits. Using the calculator, students will find that only 2310 satisfies all conditions.

3) Liz's age must be a multiple of 8, given that Mark's age is  $\frac{5}{8}$  of hers. So, Liz's age could be any of the following: 8, 16, 24, 32, 40, 48, or 64. The sum of their ages is 65, so only  $\frac{5}{8}$  of 40, or 25, would make it so that  $25 + 40 = 65$ . Since Liz's age is  $\frac{2}{5}$  of Steve's, then 40 is  $\frac{2}{5}$  of Steve's age, and Steve is therefore  $40 \times \frac{5}{2}$ , or 100 years old. For students in this category who have some background in algebra, they could set up two equations in two variables to solve for the ages of Mark and Liz, and then solve for Steve's age:  
 $M = \frac{5}{8} L$  and  $M + L = 65$ . Substitute  $\frac{5}{8} L$  for  $M$  into the second equation:  $\frac{5}{8} L + L = 65 \dots \frac{13}{8} L = 65 \dots L = \frac{(8/13)(65)}$   
So,  $L = 40$ .  
Then  $L = \frac{2}{5} S \dots 40 = \frac{2}{5} S \dots S = (40) \frac{(5/2)}{\dots} S = 100$ .  
So, Steve is 100 years old.

#### Answers

1) 4

2) 2310

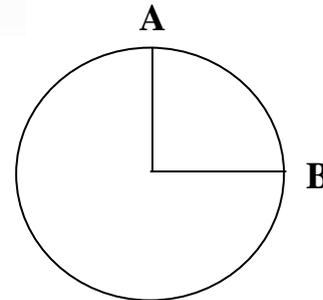
3) 100

**Category 2**  
**Geometry**  
**Meet #4 - March, 2018**

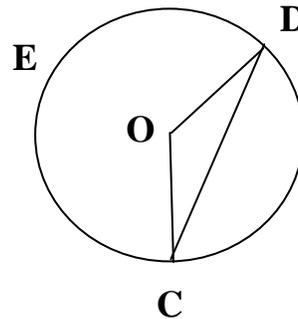


*Calculator Meet*

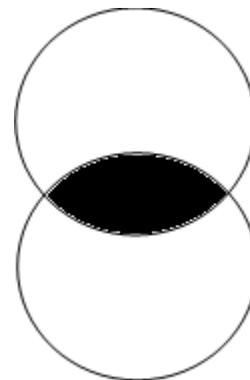
1) The radius of this circle is 12 units. The central angle is a right angle. How many units long is minor arc AB? Use  $\pi \approx 3.14$ .



2) In circle O, angle OCD measures 17 degrees. How many degrees are in major arc CED?



3) Two congruent (identical) circles intersect as shown. The radius of each circle is 16 centimeters. The two points of intersection are 16 centimeters apart. How many square centimeters are in the area of the shaded region? Use  $\pi \approx 3.1$ . If any radicals, like square roots, are used in calculations, round the decimal equivalents of those radicals to the nearest thousandth before continuing with further calculations. Round your answer to the nearest whole number of square centimeters.



**Answers**

1) \_\_\_\_\_

2) \_\_\_\_\_

3) \_\_\_\_\_

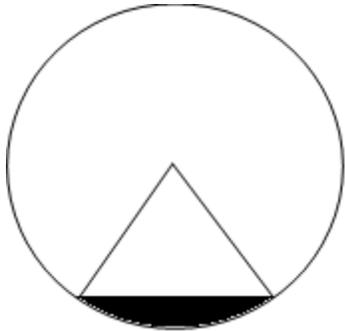
**Solutions to Category 2  
Geometry  
Meet #4 - March, 2018**

1) Arc AB =  $\frac{1}{4}(2\pi r)$   
 $\approx 0.25(2)(3.14)(12)$   
 $\approx 18.84$  units

2) Triangle COD is isosceles, as the two radii are congruent. The two acute angles each measure 17 degrees, so the central angle measures  $180 - (17 + 17)$ , or 146 degrees. Minor arc DC measures as many degrees as central angle COD, or 146 degrees. Therefore, major arc CED measures  $360 - 146$ , or 214 degrees.

3) Bisect the shaded region by connecting the two points of intersection, thus dividing the shaded region into two congruent parts. Now isolate one of the two circles with one of these shaded half-regions. Draw radii to the two endpoints of the chord, forming a sector consisting of an equilateral triangle and the shaded half-region. The measure of the central angle is, therefore, 60 degrees. The sector's area is thus  $\frac{1}{6}$  that of the circle. The altitude of the equilateral triangle can be found by dropping a perpendicular segment to the midpoint of the chord and using the Pythagorean Theorem. That altitude is  $8\sqrt{3}$ . If  $S$  = the length of one side of the equilateral triangle, then its area is  $[(S)(S)/4](\sqrt{3}) \approx [(16)(16)/4](1.732) \approx 110.848$ . The area of the shaded half-region is the difference between the area of the sector and the area of the triangle, or  $(\frac{1}{6})(\pi)(r)(r) - 110.848$  or  $132.261 - 110.848$  or 21.413. Since this is half of the shaded region, the complete shaded region's area is  $(2)(21.413)$ , or about 42.826 square centimeters. Rounding to the nearest whole number yields an answer of 43 square centimeters.

<u>Answers</u>	
1)	18.84
2)	214
3)	43



### Category 3

### Number Theory

### Meet #4 - March, 2018

### *Calculator Meet*

- 1) What is the value of the 20th number in the following arithmetic sequence?

3   7   11   15   19   . . . .

- 2) Somnia fell asleep at 9:42 P.M. She slept for 559 minutes. What time did she wake up the next morning?

- 3) The first term of an arithmetic sequence is 17 and the 60th term is 194 and the 65th term is 209. The sum of the first 60 terms is 6330. What is the sum of the 20th and 30th terms?

#### Answers

1) \_\_\_\_\_

2) \_\_\_\_\_ A.M.

3) \_\_\_\_\_

**Solutions to Category 3  
Number Theory  
Meet #4 - March, 2018**

<u>Answers</u>	
1)	79
2)	7:01
3)	178

- 1) There is a difference of 4 between any two consecutive terms. Thus, for any term  $N$ , the value of the term is  $4N - 1$ . The value of the 20th term is  $4(20) - 1$ , or 79.
- 2) Convert 559 minutes to hours and minutes, then add to 9:42 P.M. First, divide 559 by 60:  $559/60 = 9$  hours and 19 minutes. Adding that to 9:42 P.M. gives 7:01 A.M. Since the answer blank already has A.M. written, there is no need for students to write the A.M. into their answer.
- 3) The words "arithmetic sequence" indicate that there is a constant difference between any two consecutive terms. To find that difference, it is easiest to take the difference between the 60th and 65th terms and divide by the number of differences, five, that it took to reach the 65th term:  $(209 - 194) / 5 = 15 / 5$ , or 3. So, there is a constant difference of 3 between any two consecutive terms. For any  $N$ th term, the value of that term is  $3N + 14$ , given that the value of the first term is 17. So, the value of the 20th term is  $3(20) + 14$ , or 74, and the value of the 30th term is  $3(30) + 14$ , or 104. The sum of the 20th and 30th terms is  $74 + 104$ , or 178. Note: The sentence "The sum of the first 60 terms is 6330" is extraneous information, but is inserted into the problem so that students must exercise their critical reading skills to determine what data are relevant and what data are not.

**Category 4**  
**Arithmetic**  
**Meet #4 - March, 2018**



*Calculator Meet*

1) Catherine purchased a new Tesla automobile for \$78,420. When she registered the car at the DMV, she paid a sales tax of 6%. How much money did Catherine pay in sales tax?

2) The length of a rectangle was increased by 20% and its width was increased by 30%. By what percent was the original rectangle's area increased?

3) The equation to the right calculates the value  $A$ , that is the result of investing an initial amount of money,  $M$ , at an annual (yearly) rate of  $R$  for  $T$  years when the interest is compounded  $W$  times annually. Stan wanted to invest \$30,000 at his bank at an annual rate of 4.6%, compounded annually (yearly), for 15 years. Ollie insisted that he would earn more interest at his bank at an annual rate of 4%, compounded five times monthly, for 15 years. Who was correct? How much more interest was earned with the more profitable deal? Round your answer to the nearest hundred dollars. You must get both parts correct to earn credit for this problem.

$$A = M \left( 1 + \frac{R}{W} \right)^{WT}$$

**ANSWERS**

1) \$ \_\_\_\_\_

2) \_\_\_\_\_ %

3) who? \_\_\_\_\_

\$ \_\_\_\_\_

a new 2018 Tesla



**Solutions to Category 4  
Arithmetic  
Meet #4 - March, 2018**

1) Compute 6% of \$78,420:  $(0.06)(78,420)$   
 $= 4705.2$

2) If  $L$  = the original length and  $W$  = the original width, then the adjusted length and width, reflecting the increases, become  $1.2L$  and  $1.3W$  and the resulting area becomes  $(1.2L)(1.3W)$   
 $= 1.56LW$ . So, the increase in area is 56%.

3)  $A = M \left( 1 + \frac{R}{W} \right)^{WT}$   
 $= 30,000 \left( 1 + \frac{0.04}{60} \right)^{(60)(15)}$   
 $= 30,000(1 + 0.000666667)^{900}$   
 $= 30,000(1.000666667)^{900}$   
 $= 30,000(1.821754575)$   
 $= 54,652.63725$   
 $= 54,652.64$  This is how much money Ollie would have after 15 years.

Now for Stan:

$$A = M(1 + R)^T$$

$$= 30,000(1 + 0.046)^{15}$$

$$= 30,000(1.046)^{15}$$

$$= 30,000(1.963248)$$

$$= 58,897.44$$
 This is how much money Stan would have after 15 years.

The difference would be  $58,897.44 - 54,652.64$ , or \$4244.80, with Stan having earned more. Rounding the final answer to the nearest hundred yields \$4200. To earn two points credit, students must have both answers correct - Stan and 4,200. Even if students had not rounded early and had left calculations in calculator memory, the final answer would still be \$4200.

Answers

1) 4705.20  
or  
4705.2

2) 56 (%)

3) Stan  
4,200  
(Both  
must be  
correct.)



**Category 5**

**Algebra**

**Meet #4 - March, 2018**

*Calculator Meet*

- 1) At a certain time of day, a 15-foot tall tree casts a 60-foot long shadow onto the ground. How many feet long is a shadow cast by the 555-foot tall Washington Monument at the same time of day?
  
  
  
  
  
  
  
  
  
  
- 2) The sum of five consecutive odd integers is 685. What is the product of the smallest and largest of these five numbers?
  
  
  
  
  
  
  
  
  
  
- 3) Three zebras and five elephants weigh a total of 1307 stones. Four zebras and two elephants weigh a total of 758 stones. If one stone = 20 pounds, then one elephant weighs how many pounds more than one zebra?

**ANSWERS**

1) \_\_\_\_\_

2) \_\_\_\_\_

3) \_\_\_\_\_

## Solutions to Category 5

### Algebra

Meet #4 - March, 2018

### Answers

- 1) Set up a proportion, then cross products are equal:

$$\frac{\text{height of object}}{\text{length of shadow}} = \frac{15}{60} = \frac{555}{X}$$

$$15X = (60)(555)$$

$$X = 33,300 / 15$$

$$X = 2,220$$

So, the Washington Monument casts a 2,220-foot shadow.

- 2) Let  $N$  = the first odd integer. Then the next four are  $N + 2$ ,  $N + 4$ ,  $N + 6$ , and  $N + 8$

$$\text{Equation: } N + (N + 2) + (N + 4) + (N + 6) + (N + 8) = 685$$

$$5N + 20 = 685$$

$$5N = 665$$

$$N = 133$$

The smallest of the five integers is 133 and the largest is 141. Their product is  $(133)(141)$ , or 18,753.

- 3) Let  $Z$  = the weight of one zebra, in stones

$E$  = the weight of one elephant, in stones.

Now write a system of two equations in two variables, then solve for each variable. Finally, convert the number of stones to pounds by multiplying the number of stones by 20.

$$3Z + 5E = 1307$$

$$4Z + 2E = 758$$

$$12Z + 20E = 5,228$$

multiplied each side of the first equation by 4

$$12Z + 6E = 2,274$$

multiplied each side of the second equation by 3

$$14E = 2,954$$

subtracting the two equations

$$E = 211$$

divided each side by 14

$$Z = 84$$

substituting 14 for  $E$ , then solving for  $Z$ .

In pounds, one elephant weighs more than one zebra by

$$(20)(211) - (20)(84) = 4,220 - 1,680 = 2,540 \text{ pounds.}$$

1) 2,220

2) 18,753

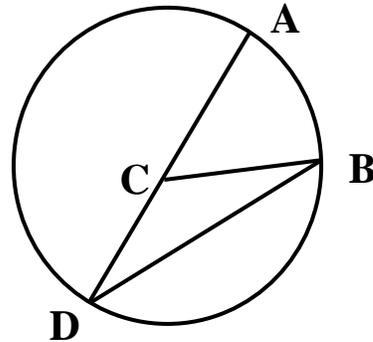
3) 2,540

**Category 6  
Team Round  
Meet #4 - March, 2018**

*Each of the following nine problems is worth four points.*

- 1) The product of two consecutive whole numbers is 6972. What is the sum of those two consecutive whole numbers?
- 2) Mya hired a limo to take her 32.8 miles from her house in Leominster to Logan Airport. The limo company charges \$3.75 per mile for each of the first five miles and \$0.85 for each quarter-mile or portion of a quarter mile after that. How much did Mya's ride to the airport cost?
- 3) When Trista was in the first grade, she was 40 inches tall. She is now 56 inches tall. By what percent did her height increase?

- 4) Point C is the center of this circle. Arc AB measures 76 degrees. AD is a diameter. How many degrees are in the measure of angle CBD?

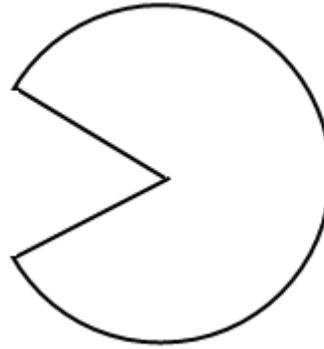


**ANSWERS**

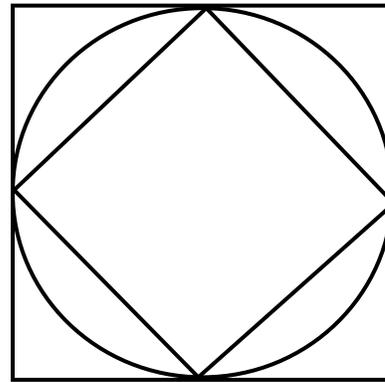
- 1) \_\_\_\_\_
- 2) \$ \_\_\_\_\_
- 3) \_\_\_\_\_ %
- 4) \_\_\_\_\_
- 5) \_\_\_\_\_
- 6) \_\_\_\_\_
- 7) \_\_\_\_\_
- 8) \_\_\_\_\_
- 9) \_\_\_\_\_

- 5) A train, travelling 72 feet per second, takes 37.5 seconds to completely enter a tunnel and another two minutes to completely pass through it. How many feet long is the tunnel?
- 6) The 12-digit number 643,409,235,81N is divisible by 12. What is the value of N?
- 7) Danny and Christopher live 405 miles apart. They made a plan to meet for lunch at a restaurant somewhere in between. Both men leave their homes at 8:00 A.M. Travelling by car, Danny averages 48 miles per hour while Christopher averages 42 miles per hour. At what time will they meet? You must include either A.M. or P.M. in your answer.

- 8) How many feet are in the perimeter of this figure consisting of a portion of a circle and two radii? The radius is 17 feet. The angle between the two radii is 40 degrees. Use  $\pi \approx 3.14$ . Round your final answer to the nearest tenth of a foot.



- 9) A square of diagonal 30 inches is inscribed inside a circle so that its vertices are four points of the circle, as shown. Another square is circumscribed about the circle so that its sides are tangent to the circle. Each vertex of the smaller square is a midpoint of a side of the larger square. How many square inches greater is the area of the larger square than the area of the smaller square? Round your answer to the nearest whole number.



**Solutions to Category 6  
Team Round  
Meet #4 - March, 2018**

**ANSWERS**

- 1) 167
- 2) 113.95
- 3) 40 (%)
- 4) 38
- 5) 8,640
- 6) 6
- 7) 12:30 P.M.
- 8) 128.9
- 9) 450

- 1) The two consecutive numbers surround the square root of 6972, or about 83.5. Multiply the whole numbers on either side to get  $83 \times 84 = 6972$ . The sum  $83 + 84 = 167$ .
- 2) Multiply  $(3.75)(5)$  to get the cost of the first five miles and then  $(0.85)(33 - 5)(4)$  to get the cost of the next 28 miles of quarter miles.  
 $(3.75)(5) + (0.85)(28)(4)$   
 $= 18.75 + 95.2$  The "33" accounts for the  
 $= \$113.95$  part of a quarter-mile.
- 3) Trista's height increased by  $56 - 40$ , or 16 inches. The ratio  $16/40$  indicates the rate of growth, or 0.4, or 40%.
- 4) The radii CB and CD help to form the isosceles triangle CBD, thus creating two congruent angles at B and D. Diameter AD creates semicircle ABD, measuring 180 degrees. Angle CDB, an inscribed angle, measures half of arc AB, or  $1/2$  of 76, or 38 degrees. Then angle CBD also measures 38 degrees.

- 5) By the time the train completely exits the tunnel, a total of 2 minutes and 37.5 seconds after it entered the tunnel, the length of the tunnel can be computed to be 72 times the number of seconds it was in the tunnel, or 120 seconds (2 minutes).  $(72 \text{ ft/sec})(120 \text{ sec}) = 8640$  feet.
- 6) The 12-digit number, to be divisible by 12, must be divisible by both 3 and 4. So, the sum of the digits must be divisible by three and the final two digits must be a multiple of 4. The sum of the digits is  $45 + N$  and the final two digits are  $1N$ . By the first condition, N could be 0, 3, 6, or 9. The only one of those four digits that makes  $1N$  a multiple of 4 is 6. Therefore, N is 6.

**SOLUTIONS TO #7-9 ARE ON THE NEXT PAGE.**

- 7) The sum of their distances is 405 miles. Let  $X$  = the number of hours it takes for the two boys to meet. Using the formula rate  $\times$  time = distance, Danny's distance is  $(48 \text{ mph})(X)$  and Christopher's distance is  $(42 \text{ mph})(X)$ . Use this equation to solve for  $X$ :

$$\begin{aligned} 42X + 48X &= 405 \\ 90X &= 405 \\ X &= 405 / 90 \\ X &= 4.5 \end{aligned}$$

Four and a half hours after they started at 8:00 A.M., they meet at 12:30 P.M.

- 8) The perimeter is  $320/360$  of the circle, plus the two radii.

$$\begin{aligned} &= \left(\frac{320}{360}\right)(2)(3.14)(17) + (2)(17) \\ &\approx (0.888888)(106.76) + (34) \\ &\approx (94.89768288) + (34) \\ &\approx 128.8976829 \\ &\approx 128.9 \text{ rounded to the nearest tenth of a foot.} \end{aligned}$$

- 9) The diameter of the inscribed square is the length of the circumscribed square. The diagonal also divides the inscribed square into two isosceles right triangles, where each leg is a side of the inscribed square. If  $X$  is the length of a side of the inscribed square, then the Pythagorean Theorem can help find  $X$ :

$$\begin{aligned} x^2 + x^2 &= 30^2 \\ 2x^2 &= 900 \\ x^2 &= 450 \quad \text{represents the area of the smaller square} \\ x &\approx 21.2132 \text{ is unnecessary (see step above)} \end{aligned}$$

Find the difference between the area of the larger square and the area of the smaller square:

$$\begin{aligned} &(30)(30) - 450 \\ &= 900 - 450 \\ &= 450 \end{aligned}$$

If the square root of 450 had been calculated and then rounded off, then the area of the smaller square would have been about  $(21.2132)(21.2132)$ , or about 499.9998... Rounded to the nearest whole number still yields an answer of 450.