IMLEM Meet #4
March, 2017

Intermediate Mathematics League of Eastern Massachusetts

This is a calculator meet!

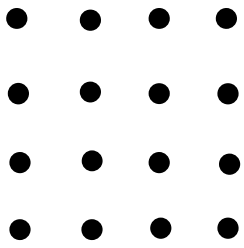
Category 1
Mystery
Meet #4 - February, 2017

Calculator Meet

1) What is the maximum (greatest) number of points of intersection that a circle and a regular 714-sided polygon can share?

2) What is the maximum number of squares, each having a perimeter of 24 inches, that can be cut from a rectangle that measures 8.5 feet by 23.5 feet? There are 12 inches in a foot.

3) Sixteen dots are arranged so that the distances between any two consecutive points, horizontally or vertically, are equal. The outer set of dots forms a square. How many squares, in all, are formed by using only the points of this grid as vertices (corners)?



Answers

1) _____

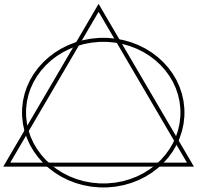
2) _____

3) _____

Solutions to Category 1
Mystery
Meet #4 - February, 2017

1) For a polygon of so many sides, one might explore the relationship between a circle and polygons of much fewer sides, as in the chart below:

# of sides	# of intersections
3	6
4	8
5	10
6	12



The max number of points of intersection is double the number of sides of the polygon. So, (2)(714) = 1428 points of intersection. A line segment can intersect a circle in three ways: no intersection, tangent or secant. So, the line can have 0,1,2 intersection points. Therefore using a maximum of 2, the maximum number of intersections an n-gon can have with a circle is 2n.

Answers

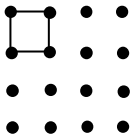
1) 1428

2) 799

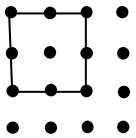
3) 20

2) A square of perimeter 24 inches has a side length of 6 inches. Each square foot can contain four such smaller squares. Calculate the number of square feet in the rectangle and multiply by four: (8.5)(23.5)(4) = 799 squares.

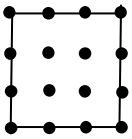
3) There are 9 one-by-one squares,



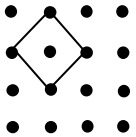
plus 4 two-by-two squares,



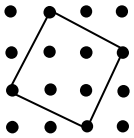
plus 1 three-by-three square,



plus 4 squares utilizing the diagonals of the one-by-ones,



plus 2 squares using the diagonals of the 1x2 rectangles in each corner,



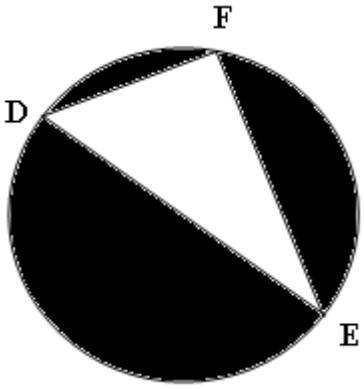
9 + 4 + 1 + 4 + 2 = 20 squares in all.

Category 2
Geometry
Meet #4 - March, 2017

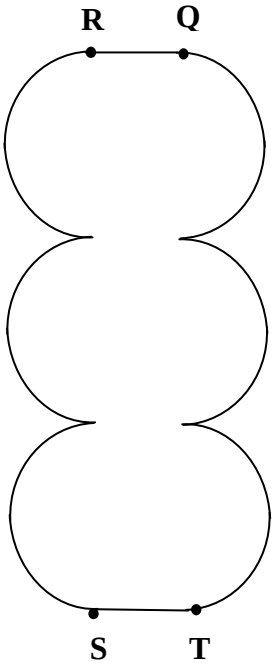
Calculator Meet

1) The diameter of this semicircle is 28 centimeters. How many square centimeters are in its area? Use $\pi \approx 3.14$. Round your final answer to the nearest tenth.

2) DE is the 25-inch diameter of this circle. FE = 20 inches. How many square inches are in the total area of the shaded regions? Use $\pi \approx 3.142$. Round your final answer to the nearest hundredth.



3) Six congruent semi-circles surround rectangle QRST. The area of the rectangle is an integer number of square meters between 160 and 168. The width of the rectangle, QR, and the length, QT, are odd prime numbers. RS > RQ. How many meters are in the perimeter of the figure? Use $\pi \approx 3.1$. Round your final answer to the nearest whole number. Note: The figure is not drawn to scale.



Answers

1) _____

2) _____

3) _____

Solutions to Category 2
Geometry
Meet #4 - March, 2017

Answers
1) 307.7
2) 340.94
3) 85

1) $A = \frac{1}{2}\pi r^2$

$A \approx \frac{1}{2}(3.14)(14)^2$

$A \approx \frac{1}{2}(3.14)(196)$

$A \approx 307.72$

$A \approx 307.7$ **rounded to the nearest tenth.**

2) Angle DFE, inscribed in a semicircle, is a right angle, so that triangle DFE is a right triangle.
Plan: a) Find the area of the triangle, b) find the area of the circle,
c) subtract the area of the triangle from the area of the circle to find the total area of the shaded regions.

a) Use the Pythagorean Theorem to find that DF = 15. Then the area of the triangle is (0.5)(15)(20) = 150.
b) Area of the circle is $A = \pi r^2$ . . . $A \approx (3.142)(12.5)^2$ . . . $A \approx (3.142)(156.25)$ . . . $A \approx 490.9375$ . . . $A \approx 490.94$ **to the nearest hundredth.**
c) 490.94 - 150 = 340.94 square inches.

3) The only number between 160 and 168 that is the product of two primes is 161. Its factors are 23 and 7. So, RQ = 7 and RS = 23. The radius of each semicircle is 23/6.
The perimeter of the figure is $2(RQ) + 6\pi r$ . . . $\approx 2(7) + 6(3.1)\left(\frac{23}{6}\right)$

$\approx 14 + (3.1)(23)$ . . . $\approx 14 + 71.3$ . . . ≈ 85.3 ≈ 85 **meters (rounded to the nearest whole number).**

Category 3
Number Theory ***Calculator Meet***
Meet #4 - February, 2017

1) The planet Mercury requires 88 days to orbit the sun. Savannah was born on Planet Earth on the 32nd day of Mercury's orbit. Her family is celebrating her 3rd birthday on Earth (when she is three years old). If Savannah has never experienced a leap year, then on which day of Mercury's orbit is her 3rd birthday? There are 365 days in an Earth year.

2) Find the value of the 438th term of the following arithmetic sequence:

63 70 77 84 91 . . .

3) Find the sum of the first 826 terms of the following arithmetic sequence:

78 87 96 105 114 . . .

Answers

1) _____

2) _____

3) _____

Solutions to Category 3
Number Theory
Meet #4 - March, 2017

Answers	
1)	71
2)	3122
3)	3,130,953

1) Savannah is 3(365) days old, or 1095. Add 1095 to 32 mod 88. First, divide 1095 by 88 to see how many Mercury orbits occur during the three years, and that is 12 with remainder 39. Add 39 to 32 to get 71 mod 88. So, Savannah's 3rd birthday falls on the 71st day of Mercury's orbit.

2)

Term #	Calculation of term value	Term value
1	63 + (0)(7)	63
2	63 + (1)(7)	70
3	63 + (2)(7)	77
4	63 + (3)(7)	84
N	63 + (N - 1)(7)	7N + 63
438	63 + (438 - 1)(7)	3122

3) Any two consecutive terms in this sequence have a difference of 9. First, find the value of the 826th term = 78 + (826 - 1)(9) = 7503. One way to find the sum of all 826 terms is to add the opposite ends (78 + 7503 = 7581) and the 2nd and 825th terms (87 + 7494 = 7581) and so on. There are 826 terms, but 826/2 pairs, or 413 pairs, that have a sum of 7581. So, the sum of all 826 terms is (413)(7581), or 3,130,953.

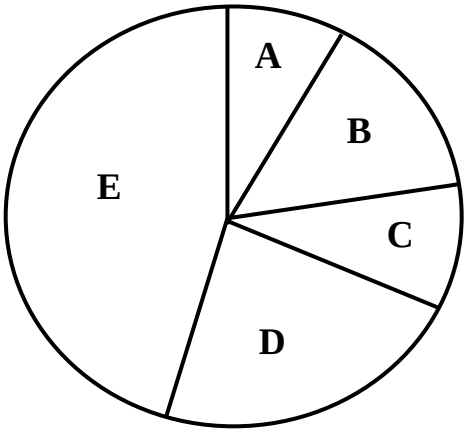
Category 4
Arithmetic *Calculator Meet*
Meet #4 - March, 2017

1) A 12-ounce sponge was soaked in water. Its weight was increased by 460%. The sponge was then wrung out (squeezed) so that its weight decreased by 80%. How many ounces does the sponge weigh now?

2) Candace counted the M&Ms in a full box as follows:

A = 19 red
B = 36 green
C = 24 blue
D = 77 orange
E = 129 yellow

What percent of all the M&Ms is green? Round your answer to the nearest whole percent.



3) The equation to the right calculates the value A, that is the result of investing an initial amount of money, M, at an annual (yearly) rate of R for T years when the interest is compounded W times annually. Emily would like to be a millionaire some day. How much money must Emily invest at an annual rate of 5.4% that compounds interest three times a month for 28 years? Round your answer either up to the next hundred dollars to assure that Emily will have at least $1,000,000 after 28 years.

$$A = M\left(1+\frac{R}{W}\right)^{WT}$$

ANSWERS

1) _____

2) _____ %

3) $ _____

Solutions to Category 4
Arithmetic
Meet #4 - March, 2017

Answers
1) 13.44
2) 13 (%)
3) 220,800

1) 12 + (4.6)(12) - (2.4)[12 + (4.6)(12)]
= 12 + 55.2 - (0.8)[12 + 55.2]
= 67.2 - (0.8)[67.2)
= 67.2 - 53.76
= 13.44 ounces

2) Divide the number of green M&Ms by the total number of M&Ms, then express the decimal answer as a percent.
36 / (19 + 36 + 24 + 77 + 129)
= 36 / 285
= 0.1263157...
= 13% when rounded to the nearest whole percent.

3) $A = M\left(1+\frac{R}{W}\right)^{WT}$ original formula

$1,000,000 = M\left(1+\frac{0.054}{(3)(12)}\right)^{(3)(12)(28)}$ substituting values, including the number T=28 for the number of years and W = 3(12) for the number of times per year the interest is compounded.

$1,000,000 = M\left(1+\frac{0.054}{36}\right)^{1008}$ evaluate

$1,000,000 = M(1+0.0015)^{1008}$ evaluate

$1,000,000 = M(1.0015)^{1008}$ evaluate

$1,000,000 = M(4.530657773)$ evaluate

$M = \frac{1,000,000}{4.530657773}$ solve for M

$M = 220,718.5027$ divide

This amount, $220,718.5027, must be rounded up to $220,800 in order to assure that Emily will have at least $1,000,000 after 28 years.

Category 5
Algebra ***Calculator Meet***
Meet #4 - March, 2017

1) Three burgers and a drink cost $18.55. Seven burgers and a drink cost $40.95. What is the cost of one drink?

2) Jason can run an average of 100 yards in 18 seconds. If he can maintain that pace, how many minutes would it take for him to run two miles? There are 1760 yards in a mile.

3) There are 2 tulips for every 5 crocuses in Alice's garden of 84 plants. Crimson's garden has 4 tulips for every 3 crocuses. If the two gardens were combined, there would be an equal number of crocuses and tulips. How many tulips are in Crimson's garden?

ANSWERS

1) $ _____

2) _____

3) _____

Solutions to Category 5
Arithmetic
Meet #4 - March, 2017

Answers	
1)	1.75
2)	10.56
3)	144

1) Let B = the number of burgers and
D = the number of drinks
3B + D = 18.55
7B + D = 40.95 Since the number of drinks is the same for each order, the difference in the number of burgers is 4 and the difference in the costs is $22.40. So, the cost of one burger is 22.4 / 4, or $5.60 and the cost of one drink is $1.75.

2) Maintaining a pace implies that the distance versus time is proportional:

$$\frac{\text{distance in yards}}{\text{time in seconds}} = \frac{100}{18} = \frac{(2)(1760)}{X}$$

Cross products are equal, so

$$100X = (18)(2)(1760)$$
$$100X = 63{,}360$$
$$X = 633.6$$

Convert 633.6 to minutes by dividing by 60, yielding an answer of 10.56 minutes.

3) Using the fact that there are 2 tulips for every 5 crocuses in Alice's garden, we can scale the total (7) by a factor of 12 for each type to make a total of 84 plants. Then Alice has (2)(12), or 24 tulips and (5)(12), or 60 crocuses, or a difference of 36 for the two types of flowers. Crimson has 4 tulips for every 3 crocuses, or a difference of 1 for the two types. Scaling that difference by a factor of 36 would give Crimson (4)(36), or 144 tulips and (3)(36), or 108 crocuses. Combining the two gardens yields a total of 24 + 144, or 168 tulips and 60 + 108, or 168 crocuses. So, Crimson has 144 tulips.

Category 6
Team Round
Meet #4 - March, 2017

Each of the following nine problems is worth <u>four</u> points.

1) Farmer Brown has only chickens and cows in his barnyard. He counts a total of 372 legs and 129 heads. How many cows are there?

2) What is the sum of the consecutive whole numbers from 126 through 361 inclusive?

3) Dogs need anywhere from 8.5 to 15 ounces of water each day for every 10 pounds of body weight. Bruiser weighs 35 pounds while Bowzer weighs 55 pounds. To the nearest tenth of a pound, what is the range of water amounts necessary, in pounds, for both dogs per week combined? (A range is the difference between the maximum and minimum values.)

4) Bruce is waiting in line to get into the movie theater. There are 78 people behind him. How many people are in front of him if there are 315 people in the line?

5) Two sides of a triangle have lengths 22 inches and 37 inches. If the third side length is a whole number of inches, then how many inches are in the maximum perimeter of the triangle?

6) Find the sum of the first eight terms of this sequence: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$

7) A painted cube with side length 1 foot is cut into 1728 cubic inches. To the nearest tenth of a percent, what percent of all the inch-cubes is not painted?

8) MX = 24 cm and is the diameter of a semicircle. A is a point somewhere along MX. How many cm are in the perimeter bounded by the shaded region consisting of three semicircles? Express your answer in terms of pi. The diagram is on the next page.

<u>ANSWERS</u>

1) _____

2) _____

3) _____

4) _____

5) _____

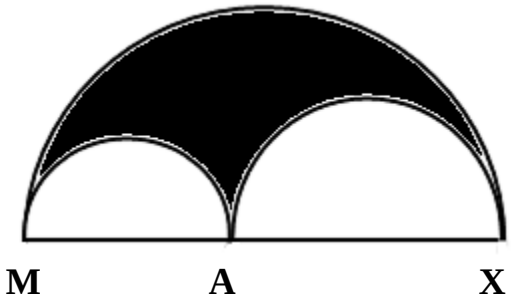
6) _____

7) _____ %

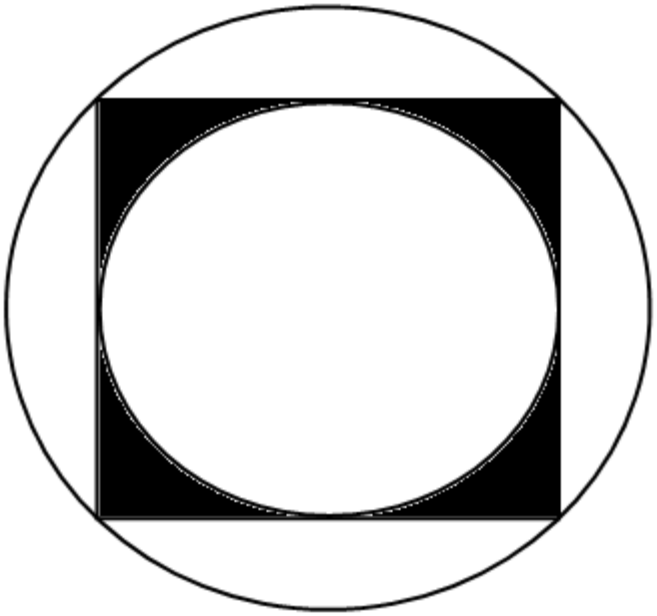
8) _____

9) _____ %

diagram for problem #8:



9) The small circle is inscribed in a square. The square is inscribed in a larger circle. The shaded region is what percent of the area of the larger circle? Round your answer to the nearest tenth of a percent.



Solutions to Category 6
Team Round
Meet #4 - March, 2017

ANSWERS

1) 57
2) 57,466
3) 25.6
4) 236
5) 117
6) $\frac{255}{256}$
7) 57.9 (%)
8) 24π
9) 13.7 (%)

1) Let X = the number of chickens
Y = the number of cows
Each chicken has two legs and one head, while each cow has four legs and one head.

$$X + Y = 129$$
$$2X + 4Y = 372$$

$$2X + 2Y = 258$$
$$2X + 4Y = 372$$

$$2Y = 114$$
$$Y = 57$$

So, there are 57 cows.

2) Total number of terms: 361 - 126 + 1, or 236 There are 236/2, or 118 sums of (126 + 361), or 487. (118)(487) = 57,466.

3) The total of the dogs' weights is 35 + 55, or 90 pounds. Scale each end of the range by a factor of 9 to get a low of (8.5)(9) = 76.5 ounces of water and a high of (15)(9) = 135 ounces of water. For a week (seven days), multiply these amounts by 7: (76.5)(7) = 535.5 ounces of water and (135)(7) = 945 ounces of water. The range = 945 - 535.5 = 409.5 ounces. The question asks for pounds, so convert 409.5 ounces to pounds by dividing by 16 = 25.59375 pounds = 25.6 pounds, when rounded to the nearest tenth of a pound.

4) 315 - (78 + 1) = 315 - 79 = 236.

5) The maximum length of the third side is 22 + 37 - 1, or 58 inches. The perimeter is 22 + 37 + 58 = 117 inches.

6) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}+\frac{1}{256}=\frac{255}{256}$

7) First find the total number of painted cubes, then subtract from 1728.

Painted cubes:
- **8 - the 8 corners**
- **120 - Along each of the 12 edges there are 10 painted cubes between the corners.**
- **600 - On each of the 6 surfaces lie an additional 10x10, or 100 painted cubes.**

Total number of painted cubes: 8 + 120 + 600 = 728.
Total number of unpainted cubes: 1728 - 728 = 1000.
The number of unpainted inch-cubes as a percent of the original cube: 1000/1728 = 0.5787 = 57.9% when rounded to the nearest tenth of a percent.

8) Regardless of where point A lies along MX, arc MA + arc AX = arc MX so that the perimeter of the shaded region is equivalent to the circumference of a circle of radius = 12.

$$C = 2\pi r = 2\pi(12) = 24\pi.$$

9) Let R = the radius of the smaller circle.
Then $R\sqrt{2}$ = the radius of the larger circle, by the Pythagorean Theorem.
One side of the square is 2R.

The area of the square is $(2R)^2$ or $4R^2$.

The area of the large circle is $\pi\left(R\sqrt{2}\right)^2$, or $2\pi R^2$.

The area of the small circle is πR^2.

The shaded area = area of square - area of small circle

$$= 4R^2 - \pi R^2.$$

The shaded area as a fraction of the large circle

$$= \frac{4R^2 - \pi R^2}{2\pi R^2}$$

$$= \frac{(4-\pi)R^2}{2\pi R^2}$$

$$= \frac{4-\pi}{2\pi}$$

$$\approx \frac{4-3.1416}{2(3.1416)} \ldots\ \approx \frac{0.8584}{6.2832} \ldots\ \approx 0.1366182 \ldots\ \approx 13.7\%.$$