IMLEM Meet #5
March, 2016

Intermediate Mathematics League of Eastern Massachusetts

This is a calculator meet!

Category 1
Mystery
Meet #5 - March, 2016

Calculator Meet

1) Jean-Claude bought a $119.84 snow board. He paid 25% right away and paid the rest in seven equal monthly payments. How much was each monthly payment?

2) In the multiplication to the right,

- **A, B, and C are different digits,**
- **A B is a two-digit number,**
- **1 C B is a three-digit number,**
- **no digit is zero, and**
- **A < B < C.**

What is the value of B ?

```
  A B
x  B
-----
1 C B
```

3) The Goldbach Conjecture states that every even number greater than two can be written as the sum of two prime numbers. For example, the number 14 can be written as the sum of two prime numbers in two different ways: 3 + 11 and 7 + 7. Notice that 11 + 3 was not included, as the order is not important. How many different ways can 60 be written as the sum of two prime numbers?

Answers

1) $ _____

2) _____

3) _____

Solutions to Category 1
Mystery
Meet #5 - March, 2016

Answers
1) 12.84
2) 5
3) 6

1) This problem can be solved without using algebra by working backwards or by solving an equation. Here, I will solve an equation. Let X = the value of each monthly installment.

$$119.84 - (0.25)(119.84) = 7X$$
$$119.84 - 29.96 = 7X$$
$$89.88 = 7X$$
$$89.88 / 7 = X$$
$$12.84 = X$$

Therefore, each monthly installment is $12.84.

2) A bit of guessing and checking, along with some critical reasoning via process of elimination, can be expedient. Since the value of B occupies every position in the units column, the only possible values of B are 5 and 6. If B = 6, then for the product to fall between 100 and 200, A must be 2. Then 26 x 6 = 156. But then it would not be true that A < B < C. However, if B = 5, then for the product to fall between 100 and 200, A must be 3. Then 35 x 5 = 175. In addition, A < B < C. Therefore, B = 5.

3) The number 60 can be written as the sum of two primes as follows:

7 + 53
13 + 47
17 + 43
19 + 41
23 + 37
29 + 31

Therefore, 60 can be written as the sum of two primes in exactly SIX ways, if order is irrelevant.

Category 2
Geometry
Meet #5 - March, 2016

Calculator Meet

1) How many rectangular blocks, each measuring 2 inches by 3 inches by 5 inches, will completely fill a rectangular cardboard box measuring 12 inches by 35 inches by 8 inches?

2) The volume of a cube with a surface area of 29,400 square units is how many times the volume of a cube whose surface area is 294 square units?

3) An NCAA regulation basketball must have a diameter of from a minimum of 9.39 inches to a maximum of 9.55 inches. One basketball of the minimum size and one of maximum size are each packed into a separate cubical box so that each ball touches all six sides of its respective container. Each box has an amount of wasted space that lies outside the basketball but inside the box. How many more cubic inches of wasted space in there in the larger box than in the smaller box? Use $\pi \approx 3.14$. Round your answer to the nearest tenth of a cubic inch. (Note: Round *only* your final answer, as any intermediate rounding - that is, rounding during other parts of solving the problem - would yield answers increasingly farther from the more precise answer.)

Answers

1) _____

2) _____

3) _____

Solutions to Category 2
Geometry
Meet #5 - March, 2016

Answers
1) 112
2) 1000
3) 20.5

1) The blocks will tessellate perfectly inside the box.
Divide the volume of the box by the volume of the block:
(12)(35)(8) / (3)(5)(2)
= 3360 / 30
= 112 blocks

2) Solving by "brute force," which is time consuming but reliable, find the length of a side of each cube, find their respective volumes, then divide the larger volume by the smaller volume.
Area of one side of larger cube = 29,400 / 6 = 4900.
one side of larger cube = the square root of 4900 = 70.
Area of one side of smaller cube = 294 / 6 = 49.
one side of smaller cube = the square root of 49 = 7.
Volume of larger cube = (70)(70)(70) = 343,000
Volume of smaller cube = (7)(7)(7) = 343.
The ratio of the two volumes is 1000 : 1.
Therefore, the larger cube has 1000 times the volume of the smaller cube.

Some students may know that if two 3-D figures are similar, then the ratio of their areas is equal to the square of the ratios of any two corresponding lengths and the ratio of their volumes is equal to the cube of the ratios of any two corresponding lengths. This knowledge produces a much quicker calculation.

3) The radius of the larger ball is 9.55 / 2 = 4.775 inches.
The radius of the smaller ball is 9.39 / 2 = 4.695 inches.
One side of each box is as long as the diameter of the ball it contains.
Volume of box - volume of ball = volume of wasted space.

larger ball: $= D^3 - \frac{4}{3}\pi R^3 = (9.55)^3 - \frac{4}{3}(3.14)(4.775)^3 =$ **415.169.**

smaller ball: $= D^3 - \frac{4}{3}\pi R^3 = (9.39)^3 - \frac{4}{3}(3.14)(4.695)^3 =$ **394.6495.**

The difference in the wasted space in the two balls is 415.169 - 394.6495 = 20.5195 cubic inches.
Rounding to the nearest tenth yields 20.5.

Category 3
Number Theory
Meet #5 - March, 2016

Calculator Meet

1) In a jar of 53 hard candies, 36 are red and 41 are sour. The candies in the jar are only just red or just sour or both red and sour. How many are just sour?

2) Set A = { 27, 29, 31, 33, 35, 37, 39 }
Set B = {30, 31, 32, 33, 34, 35, 36, 37, 38 }
What is the sum of the prime numbers in the set $A \cap B$?
(Note: $A \cap B$ is the intersection of set A and set B.)

3) Of the 460 kids at the school picnic, 30% ate a hot dog and 75% ate a hamburger. Of the kids who ate a hot dog, 65 of them also ate a hamburger. How many of the 460 kids ate neither a hot dog nor a hamburger?

Answers

1) _____

2) _____

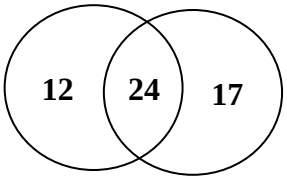
3) _____

On this date - March 31 - in 1596, Rene DesCartes, French philosopher and mathematician, was born. Our system of graphing, called the Cartesian Coordinate System, was named for him.

Solutions to Category 3
Number Theory
Meet #5 - March, 2016

Answers	
1)	17
2)	68
3)	42

1) The candies that are both red and sour: (36 + 41) - 53 = 77 - 53 = 24. This Venn diagram shows the relationship between the sets:



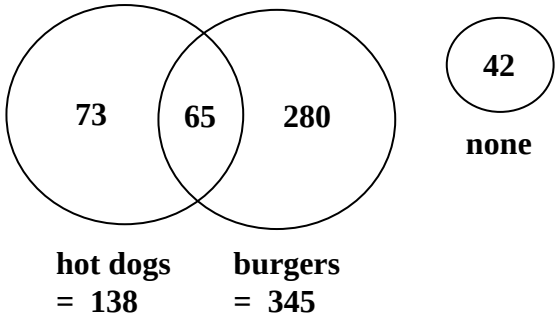
Twenty-four are both red and sour, so 36 - 24, or 12, are just red, while 41 - 24, or 17, are just sour. Therefore, 17 are just sour.

2) Hopefully, students will catch the word *prime*! The prime numbers that are common to both sets are 31 and 37, and their sum is 68.

3) The number of kids who ate hot dogs is 30% of 460, or 138. Those who ate hamburgers (burgers) is 75% of 460, or 280. The number who ate both a hot dog and a burger = 65, as given in the problem. To help fill in the blanks in the Venn diagram, those who ate just a hot dog = 138 - 65, or 73. Those who ate just a burger = 345 - 65, or 280. To calculate the number of kids who had neither a hot dog nor a burger, subtract the total number of kids who ate meat from the total number of kids who attended the picnic:

460 - (73 + 65 + 280)
= 460 - (418)
= 42

Therefore, 42 students ate neither a hot dog nor a hamburger.



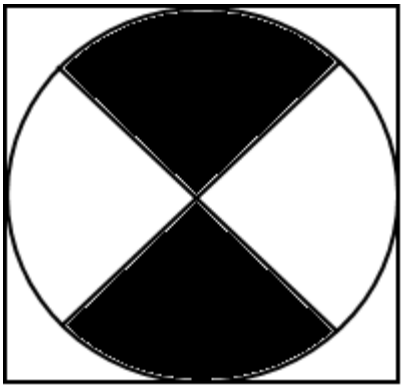
Category 4
Arithmetic
Meet #5 - March, 2016

Calculator meet

1) Our math league has one team in the state of Pennsylvania. If a letter is chosen at random from the word PENNSYLVANIA, then what is the probability that it is the letter "N?" Express your answer as a common fraction.

2) A six-sided cubical die, numbered 1 through 6 inclusive, and a ten-sided decahedral die, numbered 1 through 10 inclusive, are rolled. What is the probability that the sum of the numbers shown on the top faces is less than 8 ? Express your answer as a common fraction.

3) What is the probability that a point that lies within this square is also in the shaded area? The circle is tangent to the square (touching it on all sides) and the circle is divided into four equal sectors. Express your answer as a percent, rounded to the nearest whole percent.



ANSWERS

1) _____

2) _____

3) _____ %

Solutions to Category 4
Arithmetic
Meet #5 - March, 2016

Answers	
1)	$\frac{1}{4}$
2)	$\frac{7}{20}$
3)	39

1) There are three Ns out of the 12 letters in Pennsylvania. P = 3/12. or 1/4.

2) The ten numbers on the decahedral die are across the top of this chart, while the six numbers on the cubical die are along the left-most column. The sums are in the 10 x 6 area:

	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16

There are 21 sums that are less than eight. Therefore, the probability of rolling a sum less than eight is 21/60, or 7/20.

3) The probability that a point inside the square is also inside the shaded region is

$$\frac{area\ of\ shaded\ region}{area\ of\ square} = \frac{\frac{1}{2}\pi r^2}{(2r)^2} = \frac{(0.5)(3.14)(r^2)}{(2r)^2} = \frac{(1.57)(r^2)}{4r^2} = 0.3925 \approx 39\%.$$

Category 5
Algebra
Meet #5 - March, 2016

Calculator Meet

1) What are the two solutions to the quadratic equation below?

$$x^2 - 5x - 24 = 0$$

2) The quadratic equation $9x = 5 - 2x^2$ has two solutions. Which solution has the lesser (smaller) value?

3) The quadratic equation $H = vt - 16t^2 + h$ approximates the height in feet, H, that an object will attain after t seconds when launched from a height of h feet with an initial upward velocity (starting velocity, or speed) of v feet per second. Once the object reaches its maximum height, gravity will draw the object back toward Earth. Using a trebuchet, Neptoon launched a 30-pound watermelon upward from the edge of the deck of a large ship. The watermelon reached its maximum height and then fell back toward Neptoon but whizzed past him exactly 8 seconds after it was launched and then splashed into the sea one second later. How many feet is the deck of the ship above the surface of the sea?

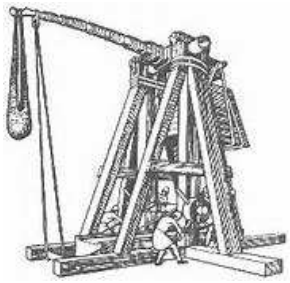
ANSWERS

1) _____

2) _____

3) _____

A trebuchet is a type of siege engine most most frequently used in the Middle Ages, often called a catapult.



Solutions to Category 5
Algebra
Meet #5 - March, 2016

Answers
1) - 3 or 8 any order
2) - 5
3) 144

1) Factor the trinomial. Set each factor to zero. Solve.

$$x^2 - 5x - 24 = 0$$
$$(x-8)(x+3) = 0$$
$$(x-8) = 0 \text{ or } (x+3) = 0$$

X = 8 or X = - 3.

2) Gather all terms onto one side of the equation, then use the strategy from solution #1.

$$9x = 5 - 2x^2$$
$$2x^2 + 9x - 5 = 0$$ **q**
$$(2x-1)(x+5) = 0$$
$$(2x-1) = 0 \text{ or } (x+5) = 0$$

X = 1/2 or X = - 5. The solution with the lesser (smaller) value is - 5.

3) Given and implied information can be substituted into the formula to acquire other values. The path of the watermelon, from its launch from the deck of the ship to the same spot 8 seconds later, completes a parabola. Substitute 0 for H and 8 for t and 0 for h to find the value of v:

$$H = vt - 16t^2 + h$$
$$0 = v(8) - 16(8)^2 + 0$$
$$0 = v(8) - 1024$$
$$8v = 1024$$
$$v = 128$$

Now substitute 9 for t (one second after 8 the 8 seconds it took for the watermelon to reach its starting height) to find the number of feet below the deck that the watermelon splashes into the sea:

$$H = (128)(9) - 16(9^2)$$
$$H = 1152 - 1296$$
$$H = -144$$

So, the watermelon splashed into the sea 144 feet below the deck of the ship.

Category 6
Team Round
Meet #5 - March, 2016

Each of the following nine problems is worth <u>four</u> points.

1) The alphabet is written over and over again until 1400 letters are written in all. What is the 1400th letter?

2) What is the probability that, in a five-child family, there are at least two girls? Assume that the probability of a girl or a boy being born is the same. Express your answer as a common fraction.

3) A plane that travels at a constant rate of 720 kilometers per hour can travel how many meters in 17 seconds?

4) Brenda spent 2/9 of her money on piano lessons and spent 1/4 of the rest of her money on donations to her favorite charities. The amount left over, $441, she deposited into her savings account at the bank. How much money did she spend on piano lessons?

<u>ANSWERS</u>

1) _____

2) _____

3) _____

4) $ _____

5) _____

6) _____

7) _____

8) _____

9) _____

5) The number 50,881 is the product of three different prime numbers, each less than 100. What is the sum of those three prime numbers?

6) What is the greatest number of points in which a circle can intersect a regular polygon of 24 sides?

7) If the time 67,980 seconds ago was 8:14 A.M., then what time is it now? You must include either A.M. or P.M.

8) If $18^{180} = (2^x)(3^y)$, then what is the value of $x + y$?

9) Consider the following sequence:
3 22 59 120 211 338 507 . . .
where 3 is the value of the first term, 22 is the value of the second term, and so on. What is the value of the 15th term ?

Solutions to Category 6
Team Round
Meet #5 - March, 2016

ANSWERS

1) V
2) $\frac{13}{16}$
3) 3400
4) 168
5) 131
6) 48
7) 3:07 A.M.
8) 540
9) 4091

1) First divide 1400 by 26 to get 53.84 . . . Then 53 full alphabets contain a total of (53)(26), or 1378 letters. 1400 - 1378 = 22, so the 22nd letter is the letter V.

2) ${}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 10 + 10 + 5 + 1 = 26$, then 26/32 = 13/16.

3) First convert 720 kilometers per hour to meters per second:

$$\frac{720km}{1hr}\cdot\frac{1000m}{1km}\cdot\frac{1hr}{60\min}\cdot\frac{1\min}{60\sec}=\frac{720,000m}{3600\sec}=\frac{200m}{1\sec}$$

Then multiply by 17 to get 3400 meters.

4) Let X = her original amount of money.

$$X - 2/9\,X - 1/4\,(7/9\,X) = 441$$
$$7/9\,X - 7/36\,X = 441$$
$$21/36\,X = 441$$
$$X = 441\,(36/21)$$
$$X = 756$$

So, Brenda started with $756 and she spent 2/9 of $756 on piano lessons, or $168.

5) The five-digit number's unit's digit is 1, so it is divisible by two-digit numbers whose unit's digit could be 1, 3, 7, or 9. Guessing and checking yields 50,881 = 17 x 41 x 73. The sum 17 + 41 + 73 = 128.

6) There is no need to actually draw a 24-sided polygon. Rather, start with polygons of much fewer sides and look for a pattern. If the polygon and the circle are drawn with the same center, it is easy to see that the number of intersection points is twice the number of sides of the polygon. So, for the 24-gon, there are (2)(24), or 48 points of intersection.

Solutions to #7-9 are on the next page.

7) Divide 67,980 by 3600 to convert to hours: 67,980 / 3600 = 18 hours and 3180 seconds. The 3180 seconds = 53 minutes. Adding 18 hours and 53 minutes to 8:14 A.M. yields 3:07 A.M.

8) The prime factorization of 18 is 2 x 3 x 3.

$$18^{180} = (2 \cdot 3 \cdot 3)^{180} = 2^{180} \cdot 3^{180} \cdot 3^{180} = 2^{180} \cdot 3^{360}$$

So, X + Y = 180 + 360 = 540.

9) To find the value of the 15th term, a lot of accurate arithmetic must take place (or some clever algebra). Most students will likely analyze the sequences of differences until the pattern looks to be predictable:

term #	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
sequence:	**3**	**22**	**59**	**120**	**211**	**338**	**507**	
diff 1:		**19**	**37**	**61**	**91**	**127**	**169**	
diff 2:			**18**	**24**	**30**	**36**	**42**	
diff 3:				**6**	**6**	**6**	**6**	

Once the student reaches the 3rd sequence of differences and sees the constant difference of 6, then s/he can work upwards to fill in the remaining sequences to the 15th term:

term #:	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
sequence:	**724**	**995**	**1326**	**1723**	**2192**	**2739**	**3370**	**4091**
diff 1:		**217**	**271**	**331**	**397**	**469**	**547**	**631**	**721**
diff 2:			**48**	**54**	**60**	**66**	**72**	**78**	**84**	**90**
diff 3:				**6**	**6**	**6**	**6**	**6**	**6**	**6**

. . . to reveal the value of the 15th term = 4091.

An algebraic approach would require the knowledge that once the 3rd sequence of differences reveals a constant, then the values of the original sequence can be expressed as following a formula for a cubic polynomial in the form $ax^3 + bx^2 + cx + d$. Four equations would be required to resolve the system to solve for a, b, c, and d. Substituting the numbers of the terms (1, 2, 3, 4) for x and then evaluating the cubic polynomial yields the following four equations:

See next page for the rest of #9 solution.

1) $a+b+c+d=3$
2) $8a+4b+2c+d=22$
3) $27a+9b+3c+d=59$
4) $64a+16b+4c+d=120$

Solving by elimination of d by subtracting equation 1 from equation 2, then 2 from 3, then 3 from 4, yields:

5) $7a+3b+c=19$
6) $19a+5b+c=37$
7) $37a+7b+c=61$

Solving by elimination of c by subtracting equation 5 from 6, and then 6 from 7, yields:

8) $12a+2b=18$
9) $18a+2b=24$

Solving by elimination of b by subtracting equation 8 from 9 yields:

10) $6a=6$ **and** $a=1$**. Then b = 3, c = 3, and d = - 4.**

Therefore, the cubic polynomial expression is x^3+3x^2+3x-4**.**
To find the value of the 15th term, substitute 15 for x:

$15^3+3(15)^2+3(15)-4$
= $3375+675+45-4$
= **4091.**