

IMLEM Meet #4
February, 2016

Intermediate Mathematics League of Eastern Massachusetts

This is a calculator meet!



Category 1

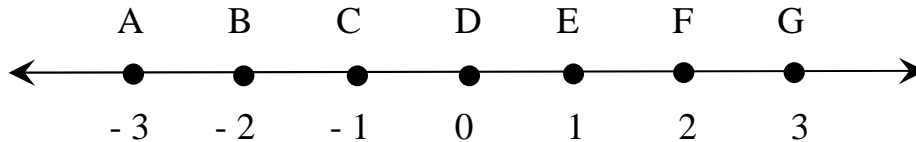
Mystery

Meet #4 - February, 2016



Calculator Meet

- 1) Let X be a non-integer that lies between A and G. Between which two consecutive named points is the value of X^3 the smallest? Give your answer as two consecutive letters in alphabetical order.



- 2) Farmer Fred wants to make a fenced-in goat pen in the shape of a rectangle that measures 136 feet by 296 feet. Each section of fence is eight feet long. Fred must first place a post into each of the four corners of the pen. The rest of the posts must be placed in a straight line on each of the four sides of the pen, eight feet apart. Posts cost \$12 each and every section of fence costs \$28. Including a 6% sales tax on the materials, what is the total cost of the goat pen?

- 3) $10^N + C$ is divisible by 3. N is a whole number greater than 50. C is a positive one-digit number. What is the sum of all possible values of C ?

Answers

1) _____

2) _____

3) _____

Solutions to Category 1

Mystery

Meet #4 - February, 2016

1) The cubes of numbers between points A and B yield the smallest values, as they are negative and their values are larger than the absolute values of the cubes of numbers in any other interval.

2) The length of the pen requires $296 / 8$, or 37 sections of fence. The width requires $136 / 8$, or 17 sections. As for the posts, there are the ones in the four corners, plus $(37 - 1)$ for the posts between the corners on each length, plus $(17 - 1)$ for the posts between the corners on each width. So, the total number of posts is $4 + 36 + 36 + 16 + 16 = 108$ and the total number of fence sections is $37 + 37 + 17 + 17 = 108$ and the total cost of the project is

$$\begin{aligned} & (\text{Cost of all posts}) + (\text{Cost of all sections of fence}) + (\text{the 6\% tax on materials}) \\ &= 12(108) + 28(108) + 0.06(\text{cost of materials}) \\ &= 1296 + 3024 + 0.06(1296 + 3024) \\ &= 4320 + 259.20 \\ &= \$ 4579.20 \end{aligned}$$

3) 10^N has only the digit 1 and a lot of zeroes after it, where the sum of its digits is 1. C must either be 2, 5, or 8 so that the sum of the digits of

$$10^N + C \text{ is either } 3, 6, \text{ or } 9. \text{ The sum of all possible values of } C \text{ is } 2 + 5 + 8 = 15.$$

Answers

1) A and B

2) 4579.20
(or 4579.2)

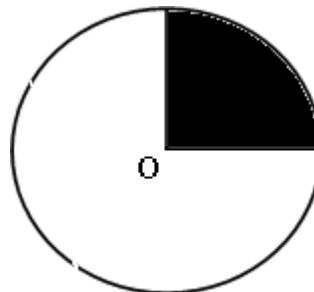
3) 15

Category 2
Geometry
Meet #4 - February, 2016

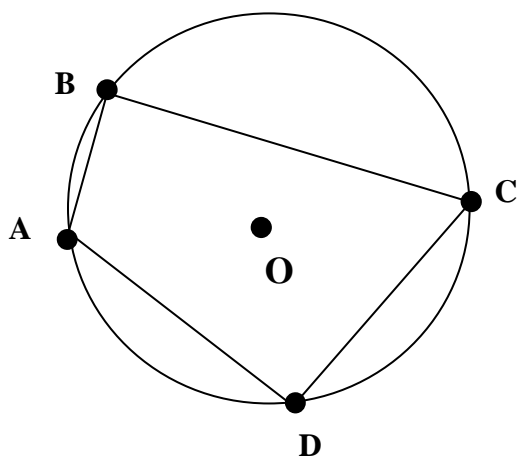


Calculator Meet

- 1) Angle O is a right angle. The radius of circle O is 8 centimeters. How many square centimeters are in the area of the shaded region? Use $\pi \approx 3.142$. Round your final answer to the nearest tenth.



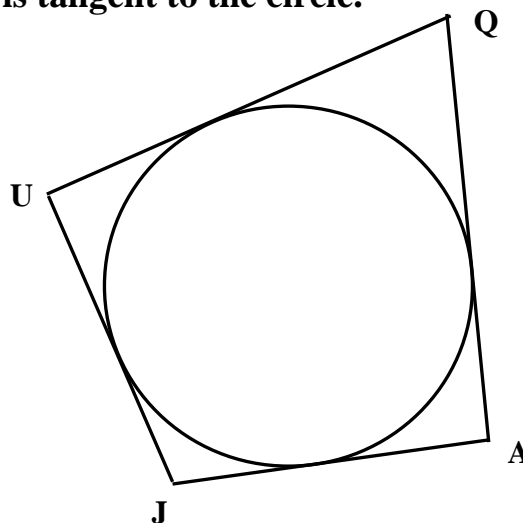
2)



- AC is the diameter of circle O .
 $AD = DC$.
 $AB = 9$ feet.
 $BC = 40$ feet.
 How many square feet are in the area of quadrilateral $ABCD$?
 Round your final answer to the nearest whole number.

- 3) $JA = 12$ decimeters. $UJ = 8$ decimeters. $QU = 18$ decimeters. How many units long is AQ ? Quadrilateral $JAQU$ is tangent to the circle.

<u>Answers</u>	
1)	_____
2)	_____
3)	_____



**Solutions to Category 2
Geometry
Meet #4 - February, 2016**

<u>Answers</u>	
1)	50.3
2)	600
3)	22

- 1) The shaded area is $\frac{1}{4}$ the area of the circle
 $= 0.25(\pi)(r^2)$
 $= 0.25(3.142)(8^2)$
 $= 0.25(3.142)(64)$
 $= 50.272$
 $=$ Rounding to the nearest tenth yields 50.3.

- 2) An angle inscribed in a semi-circle is a right angle.
 Triangle ABC is a right triangle with right angle at vertex B. Triangle ADC is isosceles as well as right. To find the area of triangle ADC, it is necessary to know the length of the diameter of the circle that is also the hypotenuse of the two right triangles. Let $x =$ the length of AD.

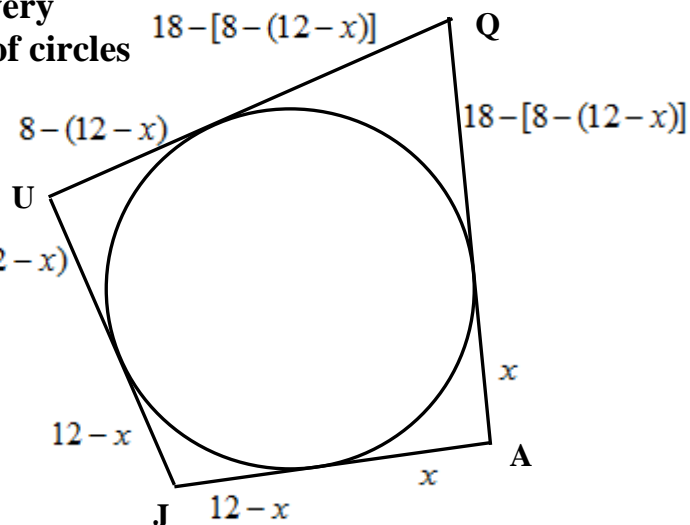
$$\begin{aligned} (AB)^2 + (BC)^2 &= (AC)^2 \\ (9)^2 + (40)^2 &= (AC)^2 \\ 81 + 1600 &= (AC)^2 \\ 1681 &= (AC)^2 \\ 41 &= AC \end{aligned}$$

$$\begin{aligned} x^2 + x^2 &= 41^2 \\ 2x^2 &= 1681 \\ x^2 &= 840.5 \\ (0.5)x^2 &= 420.25 \\ &= \text{the area of triangle ADC!} \end{aligned}$$

So, the area of quadrilateral ABCD $=$ (area of ABC) + (area of ADC)
 $= (0.5)(9)(40) + 420.25$
 $= 180 + 420.25$
 $= 600.25$
 $= 600$ when rounded to the nearest whole number.

- 3) The "around the world" strategy is very effective here, utilizing the property of circles that when two tangents are drawn to a circle from an exterior point, the tangent segments are congruent. Let $x =$ the distance from point A to the tangent point to its left. Then the following relationships exist, as we go around the circle clockwise:

$$\begin{aligned} \text{Then } AQ &= x + \{18 - [8 - (12 - x)]\} \\ &= x + \{18 - [8 - 12 + x]\} \\ &= x + \{18 - 8 + 12 - x\} \\ &= x + 22 - x = 22. \end{aligned}$$



Category 3
Number Theory
Meet #4 - February, 2016



Calculator Meet

1) Consider the sequence 7 10 13 16 19 ... where 7 is the value of the first term, 10 is the value of the second term, and so on. What is the value of the 30th term?

2) What is the sum of all whole numbers from 27 through 410 ? In other words, what is the value of

$$27 + 28 + 29 + \dots + 408 + 409 + 410 ?$$

3) Thirteen pre-schoolers sit in chairs numbered clockwise 1 through 13 that are arranged in a circle. Evan sits in chair #4 on the first day of school. The children change seats each day by rotating one seat clockwise. In which numbered seat is Evan sitting on the 180th day of school ?

Answers

1) _____

2) _____

3) _____

Solutions to Category 3
Number Theory
Meet #4 - February, 2016

<u>Answers</u>	
1)	94
2)	83,904
3)	1

- 1) The 30th term is 4 more than the 30th multiple of 3, or 94.
- 2) One simple way to compute this sum is to subtract the sum of the whole numbers 1 through 26 from the sum of the whole numbers 1 through 410.

$$\text{Sum } (1 - 26) = (26/2)(1 + 26) = (13)(27) = 351.$$

$$\text{Sum } (1 - 410) = (410/2)(1 + 410) = (205)(411) = 84,255.$$

$$84,255 - 351 = 83,904.$$

- 3) Find the value of $179 \bmod 13$. Divide 179 by 13 to get 13 with remainder 10. Since the rotation of the kids is in the same direction as the numbering of the seats, Evan will sit in seat #4 (his original seat) + 10 (the remainder) = seat #1.

Category 4
Arithmetic
Meet #4 - February, 2016



Calculator Meet

1) 24% of 40% is equal to 30% of A%. What is the value of A ?

2) The original price tag on a 1976 AMC Pacer automobile was \$3390. If a 6% sales tax is added to the price of the car, then what was the total cost of the car?

3) The equation to the right calculates the value A, that is the result of investing an initial amount of money, M, at an annual (yearly) rate of R for T years when the interest is compounded W times annually. Matthew will be able to retire once his investment grows to at least \$800,000. How much money must Matthew invest at an annual rate of 3.6% that compounds interest twice a month for 23 years? Round your answer either up or down to the nearest hundred dollars to assure that Matthew will have at least \$800,000 when he retires.

$$A = M \left(1 + \frac{R}{W} \right)^{WT}$$

ANSWERS

1) _____

2) _____

3) _____



Solutions to Category 4
Arithmetic
Meet #4 - February, 2016

<u>Answers</u>	
1)	32
2)	3593.40 zero optional
3)	349,800

- 1) $(0.24)(0.4) = 0.3N$ letting $N =$ the value of $A\%$.
 $0.096 = 0.3N$
 $N = 0.096 / 0.3$
 $N = 0.32$
 $N = 32\%$
Therefore, $A = 32$.

- 2) $(1.06)(3390) =$ the price of the car (100%) plus
the amount of sales tax (6%) = **\$3593.40.**

- 3) $A = M \left(1 + \frac{R}{W} \right)^{WT}$ original formula
- $800,000 = M \left(1 + \frac{0.036}{24} \right)^{(24)(23)}$ substituting values, including the
number 24 for the number of
half-month periods in a year
- $800,000 = M (1 + 0.0015)^{(552)}$ evaluate
- $800,000 = M (1.0015)^{(552)}$ evaluate
- $800,000 = M (2.287317)$ evaluate
- $M = \frac{800,000}{2.287317}$ solve for M
- $M = 349,754.76$ divide

This amount, \$349,754.76, must be rounded up to \$349,800 in order to guarantee at least \$800,000 in retirement money.

Category 5
Algebra
Meet #4 - February, 2016



Calculator Meet

- 1) If there are 18 minutes of commercials for the average one-hour TV show, then how many minutes of commercials should be expected during an average TV movie that lasts 210 minutes?

- 2) Working by herself, Betsy can assemble a robot in 15 hours, while Larry can do so in 18 hours. If Betsy works alone for seven hours and then Larry replaces her, then how many hours will it take for Larry to finish the job? Express your answer as a decimal.

- 3) During the month of February, Dennis enjoys making different colors of his favorite drink, Va-Va-Valentine. He mixes seltzer with organic food coloring to produce various strengths of color. He already has a container that is 10% red coloring and 90% seltzer and another container that is 30% red coloring and 70% seltzer. He would like to make 10 liters of a 15% purple drink by mixing some of the container of 10% red Va-Va-Valentine with some of the container of 30% red Va-Va-Valentine. How many liters of the 30% red Va-Va-Valentine should Dennis use? Express your answer as a decimal.

ANSWERS

1) _____

2) _____

3) _____

Solutions to Category 5
Arithmetic
Meet #4 - February, 2016

<u>Answers</u>	
1)	63
2)	9.6
3)	2.5

- 1) Let M = the number of minutes of commercials in the 210-minute TV movie. Using a proportion:

$$\frac{18}{60} = \frac{M}{210}$$

$60M = (18)(210)$ Cross products are equal.

$$60M = 3780$$

$$M = 3780 / 60$$

$M = 63.$ So, there are 63 minutes of commercials.

- 2) Use the equation: rate x time = work.
 Since Betsy can complete the job by herself in 7 hours, she works at the average rate of $1/7$ of the job per hour. Similarly, Larry works at the average rate of $1/18$ of the job per hour.
 let X = the number of hours that Larry works to complete the job.

$$\text{rate} \times \text{time} = \text{work}$$

Betsy	$1/15$	7	$7/15$
Larry	$1/18$	X	$X/18$

The work that Betsy does, added to the work that Larry does, equals one complete job:

$$7/15 + X/18 = 1$$

$$(15)(18)(7/15 + X/18) = (1)(15)(18) \quad \text{Multiply both sides by } (15)(18).$$

$$126 + 15X = 270 \quad \text{Simplify.}$$

$$15X = 144 \quad \text{Subtract 126 from each side.}$$

$$X = 144/15 \quad \text{Divide each side by 15.}$$

$$X = 9.6 \quad \text{Simplify.}$$

So, it takes Larry 9.6 hours to complete the job.

- 3) See next page for solution.

3) (# of liters of drink) x (% of coloring) = # of liters coloring

	drink	x	% color	=	coloring
10% red	R	x	0.10	=	0.10 R
30% red	10 - R	x	0.30	=	0.30 (10 - R)
blend	10	x	0.15	=	(10)(0.15) = 1.5

Utilizing just the information that is in the final column, we have that the amount of coloring in the 10% red container plus the amount of coloring in the 30% red container equals the total amount of coloring in the blend.

$$\begin{aligned} 0.10R + 0.30(10 - R) &= 1.5 \\ 0.10R + 0.30(10) - 0.30R &= 1.5 && \text{Distribute.} \\ -0.20R + 3 &= 1.5 && \text{Combine like terms.} \\ -0.20R &= -1.5 && \text{Subtract 3 from each side.} \\ R &= -1.5 / -0.20 && \text{Divide each side by -0.20.} \\ R &= 7.5. && \text{Simplify.} \\ 10 - R &= 2.5 && \text{Amount of 30\% red coloring.} \end{aligned}$$

Dennis should use 2.5 liters of 30% red coloring.

Category 6
Team Round
Meet #4 - February, 2016

Each of the following nine problems is worth four points.

- 1) The value of a two-digit number is five times the sum of its digits. What is the product of the digits?
- 2) The first sixteen terms of a sequence are
1 3 3 3 5 5 5 5 5 7 7 7 7 7 7 7
What is the sum of the reciprocals of the first thirty-six terms?
- 3) The sum of 2048 twos is equal to 2^N . What is the value of $N + 3$?
- 4) The area of a circle is decreased by 75%. Its radius was decreased by E%. What is the value of E?
- 5) How many different negative integers satisfy the inequality $x^2 < 2016$?

ANSWERS

- 1) _____
- 2) _____
- 3) _____
- 4) _____
- 5) _____
- 6) _____
- 7) _____
- 8) _____
- 9) _____

- 6) If $500 < X < 900$ and \sqrt{X} is a prime number, then what is the sum of the two possible values of X?
- 7) If $\sqrt{\frac{1}{\sqrt{N}}} = 3$ then what is the value of $12N$?
Express your answer as a common fraction.
- 8) If A is the smallest positive integer such that $84A$ is the cube of an integer, and E is the sum of the digits of A, then what is the value of E?
- 9) My favorite tarantula walks south at one kilometer per hour for one minute, then turns to walk east at 0.75 kilometer per hour for one minute. How many kilometers is the tarantula from its starting point? Express your answer as a common fraction.

Solutions to Category 6
Team Round
Meet #4 - February, 2016

ANSWERS

- 1) 20
 2) 6
 3) 15
 4) 50
 5) 44
 6) 1370
 7) $\frac{4}{27}$
 8) 18
 9) $\frac{1}{48}$

1) $10T + U = 5(T + U) \dots 10T + U = 5T + 5U$
 Then $5T = 4U$ and $T = \frac{4}{5}U$. Only 45 works because U can only be 5 so that $\frac{4}{5}U$ is a whole number. Then $T = 4, U = 5, UT = 20$.

2) The next nine numbers in the sequence are all nines and the next eleven terms are all elevens for a total of 36 terms. The reciprocal of 1 is 1, the sum of the reciprocals of the three 3s is 1, the sum of the reciprocals of the seven 7s is 1, and so on. The sum of all reciprocals is 6.

3) $(2)(2048) = 4096$, which is 2^{12} . $N = 12$, so $N + 3 = 15$.

4) Let $x =$ the rate of reduction of the radius.

$$\begin{aligned} \pi r^2 - 0.75\pi r^2 &= \pi(r - xr)^2 \\ 0.25\pi r^2 &= \pi(r - xr)^2 \\ 0.25r^2 &= (r - xr)^2 \\ 0.25r^2 &= [(r)(1 - x)]^2 \\ 0.25r^2 &= (r)^2(1 - x)^2 \\ 0.25 &= (1 - x)^2 \\ 0.5 &= 1 - x \\ 0.5 &= x \end{aligned}$$

So, $0.5 = 50\%$, so $E = 50$.

5) The negative integers are $-1, -2, -3, \dots -42, -43, -44$, as the square root of 2016 is between the integers 44 and 45. So, there are 44 negative integers that satisfy the inequality.

(See the next page for solutions to #6 - 9)

- 6) There are only two square numbers between 500 and 900 that are the squares of prime numbers, namely 529 (the square of 23) and 841 (the square of 29). The sum of the two possible values of X is 529 + 841, or 1370.

Alternatively, assume that the square root of X is prime and is between 500 and 900. Then the square root of X is between the square roots of 500 and 900, or $22.36 < \sqrt{X} < 30$. There are only two prime numbers in that range, namely 23 and 29.

- 7) Since the square root of 9 is 3, then $\frac{1}{\sqrt{N}} = 9$ and the square root of N is $1/9$. So, $N = \frac{1}{81}$ and $12N = \frac{12}{81} = \frac{4}{27}$.

- 8) Prime factor 84: $84 = 2 \times 2 \times 3 \times 7$.

For 84A to be the cube of an integer, with the smallest possible value of A, then the cube of 84A must have factors that are powers of 2, 3, and 7. Specifically, in order for 84A to be the smallest possible cube, then 84A must have factors that are each the cube of 2, 3, and 7, respectively. So, $A = 2 \times 3 \times 3 \times 7 \times 7 = 882$. If E is the sum of the digits of A, then $E = 8 + 8 + 2 = 18$.

- 9) My favorite tarantula walks $1/60$ of a kilometer going south and then $(0.75)(1/60)$, or $1/80$ of a kilometer going east. Its distance from the starting point can be acquired by using the Pythagorean Theorem.

$$\left(\frac{1}{60}\right)^2 + \left(\frac{1}{80}\right)^2 = d^2$$

$$\frac{1}{3600} + \frac{1}{6400} = d^2$$

$$\frac{6400 + 3600}{(3600)(6400)} = d^2$$

$$\frac{10,000}{23,040,000} = d^2$$

$$\frac{1}{2304} = d^2$$

$$\frac{1}{48} = d$$