IMLEM Meet #5
March/April 2013

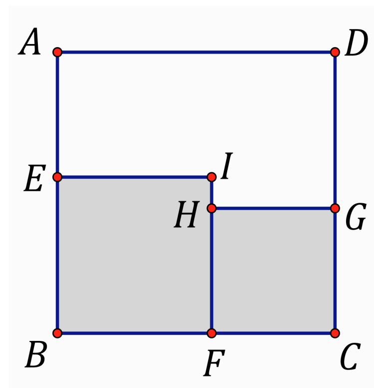
Intermediate Mathematics League of Eastern Massachusetts

Category 1
Mystery
Meet #5, March/April 2013

You may use a calculator.

1. Beth sold girl-scout cookies to some of her relatives and neighbors. She sold 4 boxes of Thin Mints for every 5 boxes of other kinds of cookies. If she sold 72 boxes in all, how many boxes of Thin Mints did she sell?

2. The area square EBFI is 25 square centimeters and the area of square HFCG is 16 square centimeters. These two squares are placed side by side in square ABCD so that points B, F and C are in a straight line. How many centimeters are there in the perimeter of the non-shaded region AEIHGD?



3. Lorene rode her bicycle along a rail trail from Appleton to Carrotville, passing through Beantown. After 30 minutes of riding, she came to a sign that said, "From here to Beantown is 3 times as far as it is from here to Appleton." She rode for another 33 miles and had already passed through Beantown when she came to a sign that said, "From here to Carrotville is 4 times as far as it is from here to Beantown." She rode another 5 hours and arrived in Carrotville. If Lorene rode at the same speed for the whole trip, how many miles did she ride in all?

Answers

1. ______________ boxes

2. _________________ cm

3. ______________ miles

Solutions to Category 1
Mystery
Meet #5, March/April 2013

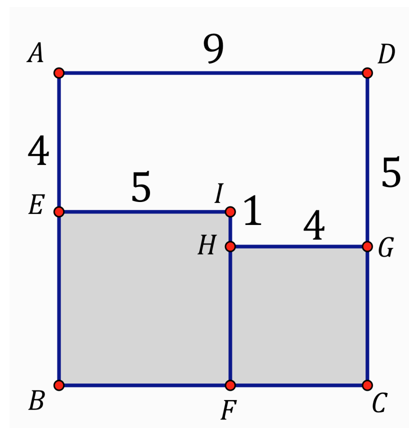
Answers

1. 32 boxes
2. 28 cm
3. 99 miles

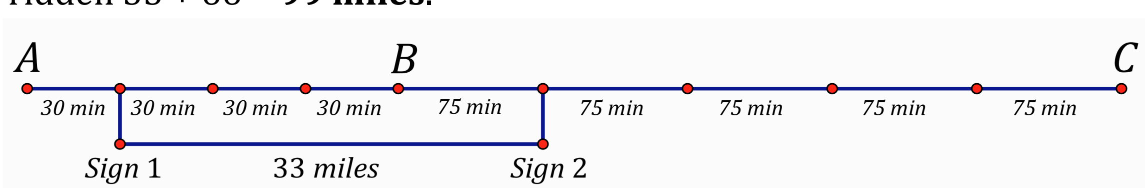
1. The ratio we are given is a “part-to-part” ratio, and we are given the “whole” number of boxes. We can see in the diagram below that a part-to-part ratio of 4 to 5 involves a total of 9 parts. If we divide the total of 72 boxes by 9, we find that each “part” (or each square in the diagram) represents 8 boxes of cookies. This means Beth must have sold 4×8 = **32 boxes** of Thin Mints and $5 \times 8 = 40$ boxes of other kinds of girl-scout cookies.

Thin Mints: □ □ □ □
Other Boxes: □ □ □ □ □

2. Square EBFI must have a side length of $\sqrt{25} = 5$ cm and square HFCG must have a side length of $\sqrt{16} = 4$ cm. If we start on the left side and work counterclockwise, we get the following lengths: 4 cm, 5 cm, 1 cm, 4 cm, 5 cm, and 9 cm. Their sum is the desired perimeter, which is **28 cm**.



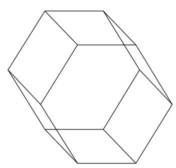
3. The first sign breaks the distance between Appleton and Beantown in a 1 to 3 ratio, which is four parts in all. It takes her 30 minutes to go from Appleton to the first sign, so it will take her $3 \times 30 = 90$ minutes to go from the first sign to Beantown. The second sign breaks the distance between Beantown and Carrotville in a 1 to 4 ratio, which is 5 parts in all. Since it takes Lorene 5 hours to ride those four parts, it must take her 5/4 hours, or 75 minutes, to go each of those parts. We can see from the diagram below that it takes her $3 \times 30 + 75 = 165$ minutes (2.75hrs), to go the 33 miles between the two signs. The total of the rest of the time is $30 + 4 \times 75 = 330$ minutes (5.5 hrs). Since this is twice the time, it must be twice the distance, which is 66 miles. Thus, Lorene must have ridden 33 + 66 = **99 miles**.

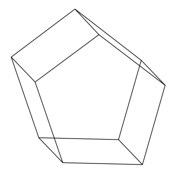


Category 2
Geometry
Meet #5, March/April 2013

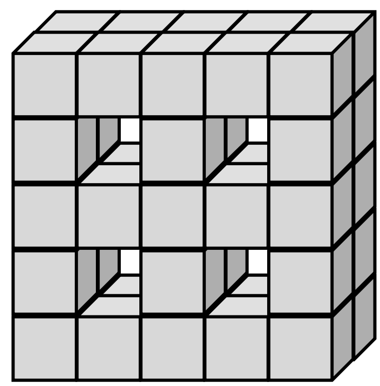
You may use a calculator.

1. How many more surface diagonals are there on a hexagonal prism than there are on a pentagonal prism?





2. Some unit cubes have been removed from a 2-by-5-by-5 structure, leaving 4 holes in the structure as shown at right. How many square units are there in the remaining surface area of the entire structure?



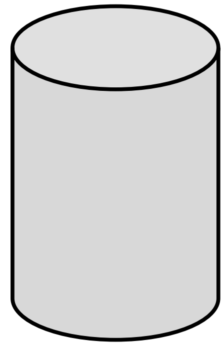
3. Mr. Peterson has a rain barrel in the shape of a cylinder. The height of the rain barrel is 42 inches and the diameter of the rain barrel is 24 inches. One cubic foot of space can hold 7.48 gallons of water. How many gallons of water does Mr. Peterson's rain barrel hold when it is full? Express your answer to the nearest whole number of gallons.

Answers

1. _________ diagonals

2. ___________ sq. units

3. ____________ gallons



Solutions to Category 2
Geometry
Meet #5, March/April 2013

Answers
1. 10 diagonals
2. 114 sq. units
3. 82 gallons

1. On the hexagonal prism, there are 9 diagonals on each of the 2 hexagonal bases and 2 diagonals on each of the 6 rectangular sides. That's 9 × 2 + 2 × 6 = 30 diagonals. On the pentagonal prism, there are 5 diagonals on each of the 2 pentagonal bases and 2 diagonals on each of the 5 rectangular sides. That's 5 × 2 + 2 × 5 = 20 diagonals. The desired difference is 30 – 20 = **10 diagonals**.

2. The surface area of a solid 2 × 5 × 5 structure would be $2(2\times5+2\times5+5\times5)=2(10+10+25)=90$ square units. When each of the holes is created, we lose 2 square units of the original surface area but create 8 more square units inside the hole. Each hole adds 8 – 2 = 6 square units, so we get 4 × 6 = 24 more square units. That brings the total surface area of the entire structure to 90 + 24 = **114 square units**.

3. The volume of a cylinder is the area of the base times the height. Mr. Peterson's rain barrel has a diameter of 24 inches, so the radius is 12 inches and the area of the base is $\pi\times12^2=144\pi$ square inches. Multiplying this by the height and using an approximation of π, we get a volume of $144\pi\times42=6048\pi\approx19000.325$ cubic inches. To convert cubic inches to cubic feet, we need to divide by 12 *three times*—once for each dimension—or by 12^3 = 1728. The result is about 10.996 cubic feet. Finally, each cubic foot is about 7.48 gallons, so Mr. Peterson's rain barrel can hold about 7.48 × 10.996 = 82.25 gallons, or **82 gallons** to the nearest whole number of gallons.

Category 3
Number Theory
Meet #5, March/April 2013

You may use a calculator.

1. Set A is the students in Mr. Taylor's homeroom and set B is the students who ride bus #1. The mathematical statement $|A \cup B| = 49$ tells us that the union of sets A and B contains 49 students. The statement $|A \cap B| = 12$ tells us that the intersection of sets A and B contains 12 students. If there are 23 students in Mr. Taylor's homeroom, how many students ride bus #1?

2. Suppose our "universal set" is all the positive integers less than or equal to 30. Suppose also that within this universal set, we have Set *P*, which is positive prime numbers, and set *E*, which is positive even numbers. Find the sum of all the elements in *U* that are not in $P \cup E$.

U = {1, 2, 3, ... 30)
P = {2, 3, 5, ... 29)
E = {2, 4, 6, ... 30)

3. Kayla plays soccer on different teams in the Fall, the Winter and the Spring. There are six girls on Kayla's Fall soccer team (including Kayla) who are also on her indoor soccer team in the Winter. These six girls are one third of the Fall team and half the Winter team. There are four girls on the Winter team (including Kayla) who also play on Kayla's Spring team. There are five girls (including Kayla) who play on the Fall and Spring teams. These five girls are one third of the Spring team. If there are three girls who play on all three teams, how many girls play on exactly one of these teams?

Answers

1. ___________ students

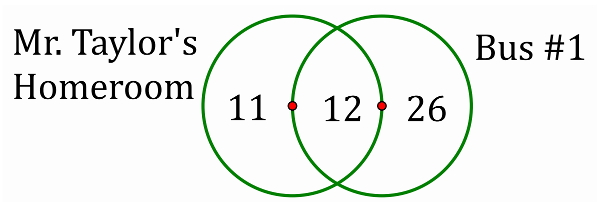
2. ____________________

3. _______________ girls

Solutions to Category 3
Number Theory
Meet #5, March/April 2013

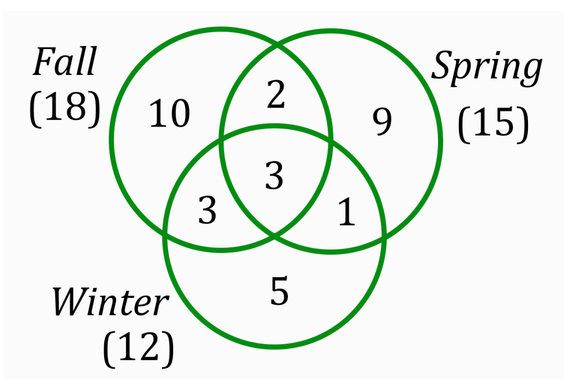
Answers
1. 38 students
2. 98
3. 24 girls

1. If we subtract the 23 students in Mr. Taylor's homeroom from the 49 students in the union, we get the 26 who ride bus #1 but are not in Mr. Taylor's homeroom. If we add the 12 students in the intersection, we get all **38 students** who ride bus #1.



2. The elements that are not in $P \cup E$ are the odd numbers less than 30 that are not prime. Their sum is 1 + 9 + 15 + 21 + 25 + 27 = **98**.

3. The Venn diagram below shows how many girls there are in each of the seven regions of the overlapping circles. We figured that there must be 6 × 3 = 18 girls on the Fall team, 6 × 2 = 12 girls on the winter team, and 5 × 3 = 15 girls on the Spring team. We placed a 3 in the very center for the 3 girls who play on all three teams. Once we had this 3 in the center, we can work out the number of girls who are on exactly two of the teams: 6 – 3 = 3 for Fall and Winter, 4 – 3 = 1 for Spring and Winter, 5 – 3 = 2 for Fall and Spring. Only then can we determine that there must be 10 girls who play only on the Fall team, 9 girls who play only on the Spring team, and 5 girls who play only on the Winter team. That's 10 + 9 + 5 = **24 girls** who play on exactly one of the teams.



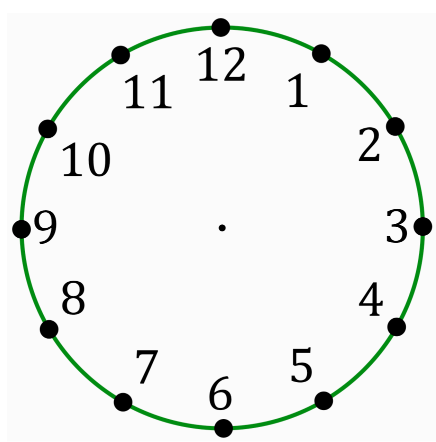
Category 4
Arithmetic
Meet #5, March/April 2013

You may use a calculator.

1. There are 3 red marbles and 1 blue marble in a bag. If 3 marbles are chosen at random, what is the probability that all three marbles are red? Express your answer as a common fraction.

2. Twenty students signed up for a weekend ski trip, but only 16 students can go. How many ways can 16 of the 20 students be chosen to go on the ski trip?

3. Cheryl created a game in which the pieces move around on the hours of a clock. Each turn, a player flips a coin and rolls two standard dice. If the coin lands on heads, he or she moves clockwise the number of spaces indicated by the sum of the numbers on the dice. If the coin lands on tails, he or she moves counterclockwise the number of spaces indicated by the sum of the numbers on the dice. If a player is on hour 10 at some point in the game, what is the probability he or she will advance to hour 2 on the next move?



Answers

1. ________________

2. ____________ ways

3. ________________

Solutions to Category 4
Arithmetic
Meet #5, March/April 2013

Answers
1. $\frac{1}{4}$
2. 4845 ways
3. $\frac{1}{9}$

1. If we choose three marbles from the bag, only one marble will be left behind. The only way to get three red marbles is to leave the one blue marble behind. The probability is thus 1/4.

2. If 16 students are to be chosen for the ski trip, then exactly 4 students won't get to go. It is easier to think about choosing the 4 students who will be left out. There are 20 possible students to choose first, followed by 19 to choose second, then 18, then 17, which would be 20 × 19 × 18 × 17 = 116,280 ways to choose, except that we don't care about the order in which a group of four students were chosen. The same group of four can been chosen in 4 × 3 × 2 × 1 = 24 different ways. In fact, the number 116,280 counts *every* group of four 24 times, so we divide by 24 to get **4845 ways** that four people can be eliminated (and thus 16 people chosen) for the ski trip. Some students will simply use the *n*C*r* function in their calculator, which computes as follows:

$$_{20}C_{16} = \frac{20!}{16!(20-16)!} = \frac{20\times 19\times 18\times 17\times 16!}{16!\times 4\times 3\times 2\times 1} = 5\times 19\times 3\times 17 = 4845$$

3. There are two ways to get from hour 10 to hour 2. The player might go clockwise by 4 hours or counterclockwise by 8 hours. To go clockwise by 4 hours, the player must flip heads and roll a sum of 4. The probability of this happening is $\frac{1}{2}\times\frac{3}{36}=\frac{3}{72}$. To go counterclockwise by 8 hours, the player must flip tails and roll a sum of 8. The probability of this happening is $\frac{1}{2}\times\frac{5}{36}=\frac{5}{72}$. Adding these two cases together, we calculate a probability of $\frac{3}{72}+\frac{5}{72}=\frac{8}{72}=\mathbf{\frac{1}{9}}$.

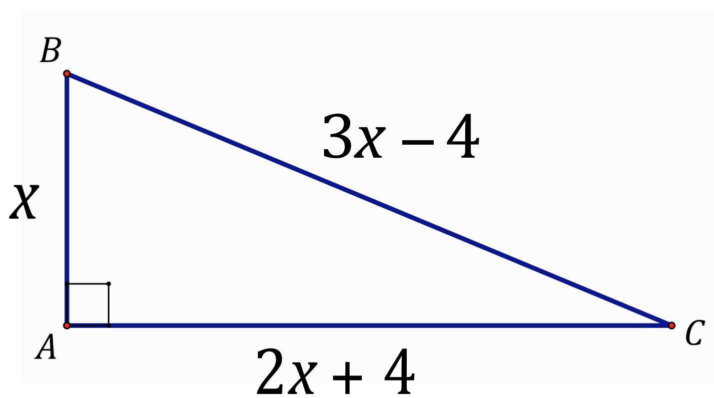
You may use a calculator.

Category 5
Algebra
Meet #5, March/April 2013

1. Terry is trying to graph a particular quadratic equation on a coordinate system. She found that the vertex of the parabola is at the point (–2.5, 90.25) and that the parabola crosses the *x*-axis at (7, 0). Find the coordinates of the other point where the parabola crosses the *x*-axis.

2. The sum of a number and its reciprocal is $2\frac{1}{30}$. What is the number if it is less than 1?

3. Triangle ABC below is a right triangle with a right angle at vertex A. How many square units are there in the area triangle ABC?



Answers

1. (______,______)

2. ____________________

3. ____________ sq. units

Solutions to Category 5
Algebra
Meet #5, March/April 2013

Answers
1. (−12, 0)
2. $\frac{5}{6}$
3. 120 sq. units

1. We should not be put off by the large *y* value of the vertex. The important part is the *x* value of the vertex, which is −2.5. The parabola has a vertical line of symmetry where $x = -2.5$. We know that one root of the equation is at (7, 0), which is 9.5 units to the right of (−2.5, 0). The other root must be 9.5 units to the left of (−2.5, 0), which is **(−12, 0)**. Note: The points where the parabola crosses the *x* axis are also known as the "roots" of the equation.

2. If we call the unknown number *x* and its reciprocal 1/*x*, we get the equation $x+\frac{1}{x}=2\frac{1}{30}$, which is equivalent to $x+\frac{1}{x}=\frac{61}{30}$. If we multiply both sides of the equation by 30*x*, we get $30x^2+30=61x$. This is a quadratic equation, so we set it equal to zero as: $30x^2-61x+30=0$. This is not easily factored, but can be solved using the quadratic formula.
A more experienced mathlete might call the unknown number *a*/*b* and its reciprocal *b*/*a*. The sum is thus $\frac{a}{b}+\frac{b}{a}=\frac{a^2}{ab}+\frac{b^2}{ab}=\frac{a^2+b^2}{ab}$, which reveals much more about the structure of the problem. Our sum is 61/30, so we need to find two factors of 30 such that the sum of their squares is 61. The obvious numbers to try are 5 and 6, and indeed $\frac{5}{6}+\frac{6}{5}=\frac{25}{30}+\frac{36}{30}=\frac{61}{30}$.
The number must be **5/6**.

3. We can use the Pythagorean Theorem to solve for *x* as shown at right. The value zero doesn't make sense, so *x* must be 10 units. The length of the other leg must be 2 × 10 + 4 = 24 units. (The hypotenuse is 3 × 10 − 4 = 26 units.) The area of the triangle is thus 10 × 24 ÷ 2 = **120 square units**.

$$x^2+(2x+4)^2=(3x-4)^2$$
$$x^2+4x^2+16x+16=9x^2-24x+16$$
$$-4x^2+40x=0$$
$$-4x(x-10)=0$$
$$x=0 \text{ or } x=10$$

Category 6
Team Round
Meet #5, March/April 2013

You may use a calculator.

1. A piano teacher must choose four of her six seventh graders and five of her seven eighth graders to perform in a piano recital. She must then determine the order in which these nine students will perform. If the seventh graders must all perform before any of the eighth graders, how many possible orders are there for the recital program?

2. Find all solutions to the equation $(2x-5)(2x-9)+(x-5)(2x-5)=0$. Express your answers as common fractions.

3. Suppose you have a lot of unit cubes and you glue some of them together to make a larger cube that is 8 by 8 by 8 of these unit cubes. If you hold this larger cube in front of you and rotate it any way you want, what is the greatest number of the unit cubes you can see at any given moment?

4. The set of children in a family is {Abe, Beth, Carl, Doug, Ellie, Fred}, where Beth and Ellie are girls and the others are boys. What fraction of all possible subsets of this set contains at least one girl? Express your answer as a common fraction.

5. Jihoon flipped a coin 8 times in a row. If it is known that he did not flip two consecutive heads, what is the probability that he flipped exactly 3 heads? Express your answer as a common fraction.

6. Evaluate the expression below, using the answers from questions 1 through 5. For *B*, use the product of all the solutions you found.

$$\frac{\sqrt{7AC}}{BDE}$$

Answers

1. _________ orders = *A*
2. ________________ ≠ *B*
3. _________ cubes = *C*
4. ________________ = *D*
5. ________________ = *E*
6. ________________

Solutions to Category 6
Team
Meet #5, March/April 2013

Answers

1. 907,200 orders
2. $\frac{5}{2}, \frac{14}{3}$
3. 169 cubes
4. $\frac{3}{4}$
5. $\frac{4}{11}$
6. 10,296

1. Since the order of the performances matters, these are permutations, not combinations. The calculations are shown below.

$${}_6P_4 \times {}_7P_5 = (6 \times 5 \times 4 \times 3) \times (7 \times 6 \times 5 \times 4 \times 3) = \mathbf{907{,}200}$$

2. If we expand both products and collect like terms, we will get a quadratic equation. On the other hand, if we notice the common factor of $2x - 5$, we can rewrite the left side of the equation as shown below. The result is a quadratic equation that is already factored into a product of two binomials.

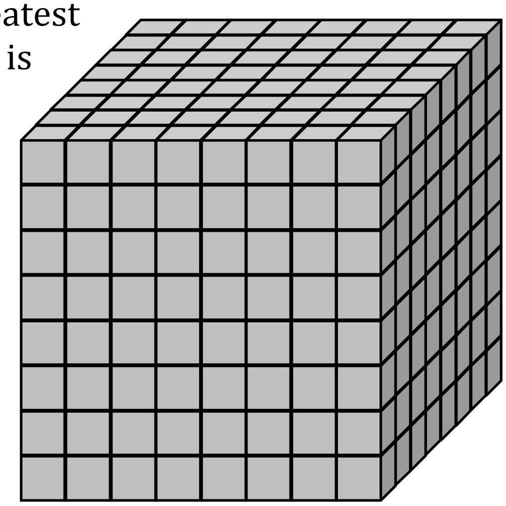
$$(2x-5)(2x-9)+(x-5)(2x-5)=0$$

$$(2x-5)\left[(2x-9)+(x-5)\right]=0$$

$$(2x-5)(3x-14)=0$$

$$x=\mathbf{\frac{5}{2}} \text{ or } x=\mathbf{\frac{14}{3}}$$

3. No matter how you rotate a cube, the greatest number of faces you can see is three. There is a danger of overcounting the cubes that are along shared edges. If we count all 8 × 8 = 64 cubes on the front face, we can then count the 7 × 8 = 56 cubes on the top that are behind to common edge. Finally, we count the 7 × 7 = 49 cubes on the right side that are not along either of the common edges. That makes 64 + 56 + 49 = **169 cubes** that can be seen at the same time.



4. If we remove the two girls, we have the set {Abe, Carl, Doug, Fred}. There are $2^4 = 16$ subsets of these four boys, including the empty set. If we subtract these 16 subsets from the $2^6 = 64$ subsets for all six children, the other 64 – 16 = 48 subsets must contain at least one girl. The desired fraction is 48/64 = **3/4**.

5. Jihoon could have flipped 0, 1, 2, 3, or 4 heads. (He could not have flipped more than 4 heads without getting consecutive heads.) We need to count each of these situations separately, since the sum of these possibilities will be the denominator of our probability. The number of ways to flip three heads without consecutive heads will be the numerator. In each case below, we are thinking about a string of T's and asking how many ways we can insert a number H's on the blanks between the T's. From the table, we see that there are 1 + 8 + 21 + 20 + 5 = 55 ways to flip 8 coins without getting consecutive heads. Twenty of these 55 ways have exactly 3 heads, so the probability is 20/55 = **4/11**.

Number of Heads	String of Tails and spaces	Number of Ways
0	_T_T_T_T_T_T_T_T_	${}_9C_0 = 1$
1	_T_T_T_T_T_T_T_	${}_8C_1 = 8$
2	_T_T_T_T_T_T_	${}_7C_2 = \frac{7\times 6}{2\times 1} = 21$
3	_T_T_T_T_T_	${}_6C_3 = \frac{6\times 5\times 4}{3\times 2\times 1} = 20$
4	_T_T_T_T_	${}_5C_4 = \frac{5\times 4\times 3\times 2}{4\times 3\times 2\times 1} = 5$

6. Using the values for *A* through *E* in the expression given, we get:

$$\frac{\sqrt{7AC}}{BDE} = \frac{\sqrt{7\times 907200\times 169}}{\left(\frac{5}{2}\times\frac{14}{3}\right)\times\frac{3}{4}\times\frac{4}{11}} = \frac{32760}{\frac{35}{11}} = \frac{32760}{1}\times\frac{11}{35} = \mathbf{10{,}296}$$