IMLEM Meet #4
February 2013

Intermediate Mathematics League of Eastern Massachusetts

IMLEM Meet #4
February 2013

Statistics and notes – not part of the original meet

Scheduled Meet Date	Feb. 14, 2013	
Number of Teams Competing	75	
Average Team Score	104	
Average Individual Score	9.0	(out of 18)

Category	1 Myst	2 Geom	3 NumTh	4 Arith	5 Alg
Number of Regulars Competing in This Category	437	436	432	434	437

Percent of Regulars with each possible score in the category:

0	17%	44%	11%	20%	17%
2	29%	26%	28%	32%	23%
4	45%	16%	31%	33%	26%
6	9%	14%	30%	15%	35%

Percent of Team Rounds with each possible score:

0	6	12	18	24	30	36
16%	7%	15%	15%	36%	1%	11%

Informal targets for evaluating contest difficulty level:

1) Target Team Score: 100
   →104 Good
2) "Not too hard"-- Target Percentage of Regulars in hardest category of meet with zero points: 20%
   →Geometry 44%
3) "Not too easy" -- Target Percentage of perfect categories: 15%
   →Number Theory 30%, Algebra 35%
4) "Team not too hard" -- Percentage of zero-point Team Rounds: 5%
   →16% - hard
5) "Team not too easy" -- Target Percentage of perfect Team Rounds: 10%
   →11% - good.

Category 1 *You may use a calculator.*
Mystery
Meet #4, February 2013

1. In a certain family, each girl has the same number of brothers as sisters, but each boy has three times as many sisters as brothers. How many kids are there in this family?

2. Below is a table of apothecaries' weights. (These are real units of measure!) If Ella has 3 ounces, 2 drams and 1 scruple of silver, how many grains of silver does she have?

20 grains = 1 scruple
3 scruples = 1 dram
8 drams = 1 ounce

3. When Mr. Dimension, a math teacher, retired, he gave all his math books to his colleagues Mrs. Abacus, Mr. Binary, and Ms. Cardinal. First Mrs. Abacus took 1/3 of the books plus 1/3 of a book. Next Mr. Binary took 1/2 of the remaining books plus 1/2 of a book. Finally, Ms. Cardinal took the last 13 books. None of the books were cut up in the process of sharing. How many books did Mr. Dimension give to his colleagues?

Answers

1. _____ kids
2. _____ grains
3. _____ books

Solutions to Category 1
Mystery
Meet #4, February 2013

Answers
1. 5 kids
2. 1580 grains
3. 41 books

1. There are **5 kids** in the family, 2 boys and 3 girls. Each girl has 2 brothers and 2 sisters, and each boy has 1 brother and 3 sisters.

2. Ella's 3 ounces of silver are equivalent to $3 \times 8 = 24$ drams. Adding these 24 drams to her 2 drams of silver, she has $26 \times 3 = 78$ scruples. Adding thse 78 scruples to her 1 scruple of silver, she has $79 \times 20 =$ **1580 grains** of silver.

3. Let's work backwards to figure this out. When Ms. Cardinal took the last 13 books, this must have been half a book less than half of the number of books Mr. Binary saw before he took his. He must have seen $\left(13+\frac{1}{2}\right)\times 2 = 27$ books. These 27 books must have been one third of a book less than the two thirds of the books that remained after Mrs. Abacus took hers. She must have seen $\left(27+\frac{1}{3}\right)\times\frac{3}{2}=\frac{82}{3}\times\frac{3}{2}=\mathbf{41}$ books, which is the number of books Mr. Dimension gave to his colleagues.

Check:

Mrs. A takes $\frac{1}{3}\times 41+\frac{1}{3}=\frac{41}{3}+\frac{1}{3}=\frac{42}{3}=14$ books, leaving 41 – 14 = 27 books.

Mr. B takes $\frac{1}{2}\times 27+\frac{1}{2}=\frac{27}{2}+\frac{1}{2}=\frac{28}{2}=14$ books, leaving 27 – 14 = 13 books.

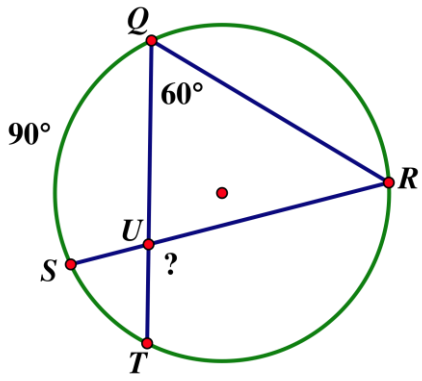
Ms. C takes the last 13 books. It works!

Category 2
Geometry
Meet #4, February 2013

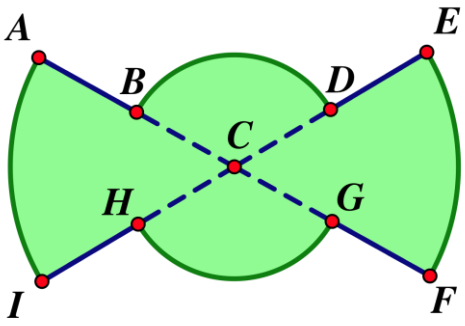
You may use a calculator.

1. A car turns 180 degrees on an arc of a circle. The tires on the inside of the turn are 20 feet from the center of the arc and the tires on the outside of the turn are 25 feet from the center of the arc. How many feet farther do the outside tires travel than the inside tires? Use 3.14 for π and round your answer to the nearest whole number of feet.

2. In circles, the measure of an inscribed angle is equal to half the measure of the intercepted arc. If the measure of arc QS is 90° and the measure of angle RQT is 60°, as shown in the figure, how many degrees are there in the measure of angle RUT?



3. The figure below was created from two concentric circles with centers at point C. Line segments AF and EI are bisected by point C, and each of the resulting segments is then bisected at points B, D, G, and H, as shown. If segment AB is 2 cm and angle BCD is 120 degrees, how many square centimeters are there in the entire figure? Use 3.14 for pi and express your answer to the nearest tenth of a square centimeter.



Answers

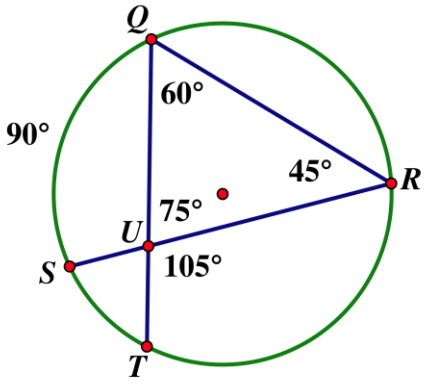
1. _____ feet
2. _____ degrees
3. _____ sq. cm

Solutions to Category 2
Geometry
Meet #4, February 2013

Answers
1. 16 feet
2. 105 degrees
3. 25.1 sq. cm.

1. The circumference of an entire circle is the diameter times π, so half that amount is the radius times π. The tires on the outside of the turn travel 25π feet and the tires on the inside of the turn travel 20π feet. The difference is 25π – 20π = 5π, which is about 15.7 feet, or **16 feet** to the nearest foot.

2. Angle QRS intercepts arc QS, so it must be 90 ÷ 2 = 45 degrees. Since triangles have an angle sum of 180 degrees, the measure of angle QUR must be 75 degrees. Finally, angle RUT is supplementary to QUR, so it must measure 180 – 75 = **105 degrees**.



3. The area of a circle is the square of the radius times π. Our picture can be thought of as 2/3 of a circle with radius 2 cm and 2/6 = 1/3 of a circle with radius 4 cm. The area is thus

$\frac{2}{3}\times\pi\times 2^2+\frac{1}{3}\times\pi\times 4^2=\frac{8}{3}\pi+\frac{16}{3}\pi=\frac{24}{3}\pi=8\pi\approx$ **25.1** square centimeters.

Category 3 *You may use a calculator.*
Number Theory
Meet #4, February 2013

1. Find the 111^{th} term in the arithmetic sequence below.

38, 44, 50,…

2. Find the sum of the first 8 terms in the geometric sequence below.

8, 24, 72, …

3. A mathematician grandmother arranged a Yankee gift swap among her 12 grandchildren in the following way. First she arranged their names in order from oldest to youngest and assigned them each a number from 1 to 12. She then determined that each grandchild n would buy a gift for grandchild $2^n \pmod{13}$. For example, grandchild 4 would buy a gift for grandchild 3, since $2^4 = 16 = 3 \left(\mathrm{mod}\, 13\right)$. Find the number of the grandchild who would buy a gift for grandchild 5.

Answers

1. _____
2. _____
3. grandchild _____

Solutions to Category 3
Number Theory
Meet #4, February 2013

Answers
1. 698
2. 26,240
3. grandchild 9

1. Consecutive terms in an arithmetic sequence have a "common difference." The common difference between the terms in this sequence is 44 – 38 = 50 – 44 = 6. If we want to get from the first term to the 111th term, we have to add six 110 times. The result is 38 + 110 × 6 = 38 + 660 = **698**. Some students may prefer to place a zeroth term of 32 before the first term 38. In that case, one would add 111 × 6 = 666 to 32 to get the same result of 698 for the 111th term.

2. Consecutive terms in a geometric sequence have a "common ratio." In this sequence, the common ratio is 24 ÷ 8 = 72 ÷ 24 = 3. To find the sum of the first 8 terms, we can actually calculate those 8 terms and keep a running sum, as shown in the table below.

Count	1	2	3	4	5	6	7	8
Term	8	24	72	216	648	1944	5832	17496
Sum	8	32	104	320	968	2912	8744	**26240**

Alternatively, we can use the formula below, with $a = 8$, $r = 3$, and $n = 7$.

$$S = a\left(\frac{1-r^{n+1}}{1-r}\right) = 8\left(\frac{1-3^8}{1-3}\right) = 8 \times \frac{-6560}{-2} = 26{,}240$$

3. We are trying to solve the equation $2^n = 5 \pmod{13}$ for n. We can just keep doubling, dividing by 13 whenever the result is greater than 13 and looking all the while for a remainder of 5. The table below shows the assignments for all 12 grandkids. We see that **grandchild 9** buys a present for grandchild 5.

n	1	2	3	4	5	6	7	8	**9**	10	11	12
$2^n \pmod{13}$	2	4	8	3	6	12	11	9	5	10	7	1

Category 4 *You may use a calculator.*
Arithmetic
Meet #4, February 2013

1. This year, an all-day adult lift ticket at Ski Mountain costs $78, which is 4% more than last year. How much did an all-day adult lift ticket cost last year?

2. The original price of a winter coat was $60 before it was marked down by 30% for a sale. Mary bought the coat during the sale but she also used a 10% off coupon. How much did she pay for the coat? Express your answer in dollars to the nearest hundredth of a dollar.

3. Allan deposited $10,000 into a savings account that paid 3% annual interest compounded monthly. Izak deposited $10,000 into a savings account that paid 4% annual interest compounded monthly. Neither of them withdrew any money from his account. After one year, how much more money did Izak have in his account than Allan had in his account? Remember that, each month, interest will be paid on the balance at 1/12 of the annual rate. Express your answer in dollars to the nearest hundredth of a dollar.

Answers

1. $ _____
2. $ _____
3. $ _____

Solutions to Category 4
Arithmetic
Meet #4, February 2013

Answers	
1.	$75 or $75.00
2.	$37.80
3.	$103.26

1. The price of $78 represents 104% of the previous year's price. To find the previous year's price, we simply divide 78 by 1.04, which is **$75**.

2. When a price is marked down by 30%, it means the new price is 100 – 30 = 70% of the old price, so the sale price of the coat was 60 × 0.7 = $42. Mary also used a 10% off coupon, so she paid 100 – 10 = 90% of the sale price, which was 42 × 0.9 = **$37.80**.

3. We can calculate the year-end balances directly as follows:

$$10000\times\left(1+\frac{0.03}{12}\right)^{12}\approx 10304.16 \text{ and}$$

$$10000\times\left(1+\frac{0.04}{12}\right)^{12}\approx 10407.42$$

After one year, Allan has 10,407.42 – 10,304.16 = **$103.26** more than Izak. Alternatively, we could keep track of the balances at the end of each month, as shown in the table below.

Month	Allan's Balance	Izak's Balance
0	$10,000.00	$10,000.00
1	$10,025.00	$10,033.33
2	$10,050.06	$10,066.78
3	$10,075.19	$10,100.33
4	$10,100.38	$10,134.00
5	$10,125.63	$10,167.78
6	$10,150.94	$10,201.67
7	$10,176.32	$10,235.68
8	$10,201.76	$10,269.80
9	$10,227.26	$10,304.03
10	$10,252.83	$10,338.38
11	$10,278.46	$10,372.84
12	$10,304.16	$10,407.42

Category 5
Algebra
Meet #4, February 2013

You may use a calculator.

1. Abby weighs 9 pounds less than twice as much as her younger sister Bridget. If they weigh 102 pounds together, how many pounds does Abby weigh?

2. Benny and Lenny live 1 mile from their school. Benny walks to school at an average speed of 3 mph and his brother Lenny rides his bike at an average speed of 10 mph. If Lenny wants to arrive at school 2 minutes before Benny, how many minutes after Benny should he leave the house?

3. If 7 thneed-makers can make 444 thneeds in 6 hours and 10 minutes, how many thneeds can 10 thneed-makers make in 4 hours and 40 minutes?

Answers

1. _____ pounds
2. _____ minutes
3. _____ thneeds

Solutions to Category 5
Algebra
Meet #4, February 2013

Answers
1. 65 pounds
2. 12 minutes
3. 480 thneeds

1. We will let *A* be Abby's weight and *B* be Bridget's weight. Translating the English to algebra, we get $A = 2B - 9$ and $A + B = 102$. We can substitute the value of *A* from the first equation into the second equation and solve for *B* as shown below.

$$2B - 9 + b = 102$$
$$3B - 9 = 102$$
$$3B = 111$$
$$B = 37$$

The value of *A* must be 102 – 37 = **65 pounds**. We can also check that $2 \times 37 - 9 = 74 - 9 = 65$ is the same, which it is.

2. Since Benny can walk 3 miles in 1 hour, it will only take him 1/3 of an hour, or 20 minutes, to walk the 1 mile to school. Similarly, it will take Lenny 1/10 of an hour, or 6 minutes, to ride the 1 miles to school. If Lenny wanted to arrive at the same time as Benny, he could leave 20 – 6 = 14 minutes later. Since he wants to arrive at school 2 minutes before Benny, Lenny should leave 14 – 2 = **12 minutes** after Benny.

3. First let's convert the times to minutes: 6 hrs. 10 min. = 370 min. and 4 hrs. 40 min. = 280 min. For now, let's just stick with 7 thneed-makers and see how many thneeds they could make in 280 minutes. We could set up a proportion: $\frac{444 \text{ thneeds}}{370 \text{ minutes}} = \frac{x \text{ thneeds}}{280 \text{ minutes}}$. Solving for *x*, we get $x = \frac{444 \times 280}{370} = 336$ thneeds. If we divide this by 7, we find that each theed-maker must make 336 ÷ 7 = 48 thneeds in 280 minutes. Therefore, 10 thneed-makers would make 10×48 = **480 thneeds** in 280 minutes.

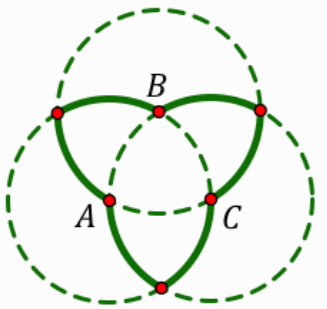
Category 6
Team Round
Meet #4, February 2013

You may use a calculator.

1. A triangle's base is increased by 20% and its height is decreased by 20%. If the area of the original triangle is 375 square cm, how many square cm are in the area of the triangle after the changes are made?

2. Find the fourth of six consecutive even integers if the sum of the least two integers is five times the greatest of the six integers.

3. Each of the points A, B, and C is the center of a circle and a point on the other two circles. If the radius of each circle is 10 cm, how many cm are there in the length of solid path in the figure? Round your answer to the nearest tenth of a centimeter.



4. What is the sum of the least and the second-least natural numbers n that are solutions to the equation $7n = 5 \pmod{13}$?

5. In Dr. Seuss's book <u>The Cat in the Hat Comes Back</u>, the Cat in the Hat takes off his hat to reveal a smaller Cat in the Hat named A. This cat takes off his hat to reveal a smaller Cat in the Hat named B, and so on. Suppose each cat is 4 times the height of his own hat and the height of each cat plus his hat is equal to the height of the previous cat's hat. If the original Cat in the Hat is 6 feet tall, how tall is the Cat in the Hat named C? Express your answer as a decimal to the nearest thousandth of an inch.

Answers

1. _____ sq. cm = A
2. _____ = B
3. _____ cm = C
4. _____ = D
5. _____ in. = E
6. _____

6. Evaluate the expression below, using the answers from questions 1 through 5. Round your answer to the nearest tenth.

$$\sqrt{0.1A+B+10C+100D+1000E}$$

Solutions to Category 6
Team Round
Meet #4, February 2013

Answers
1. 360 sq. cm
2. –10
3. 62.8 cm
4. 33
5. 0.576 inches
6. 67.3

1. If the base of the triangle is increase by 20%, then the new base is 120% of the old base, or 1.2 times as long. If the height is decreased by 20%, then the new height is 80% of the old height, or 0.8 times as long. The area of the triangle after the changes is $0.8 \times 1.2 = 0.96$ of the old area, so it's $0.96 \times 375 =$ **360 sq. cm**.

2. If a is the least of the integers, then the six integers can be represented as a, $a+2$, $a+4$, $a+6$, $a+8$, and $a+10$. The sum of the least two is said to be five times the greatest of the integers, so we can write the equation $a + (a+2) = 5(a+10)$. Simplifying this, we get $2a + 2 = 5a + 50$. If we subtract both $2a$ and 50 from both sides, we get $-48 = 3a$, which means that $a = -16$. The six integers are -16, -14, -12, -10, -8, and -6. We want the fourth of the six integers, so **–10** is our answer.

3. Each arc is one sixth of a circle and there are six arcs, so the length of the path is equal to the circumference of one full circle of radius 10 cm, which is $2 \times \pi \times 10 \approx 20 \times 3.14 =$ **62.8 cm**.

4. We could list some multiples of 7 and hope to notice one that is exactly 5 more than a multiple of 13. Alternatively, we could list some numbers that are 5 more than a multiple of 13 and hope to notice a multiple of 7. Since we are more likely to recognize a multiple of 7, the second plan is better. Here's our list: 5, 18, 31, 44, 57, *70*,… *161*. The least value of n is 10, since $7 \times 10 = 70$. Notice that we take a jump of $7 \times 13 = 91$ on our list. The next number that is divisible by 7 can only happen after we add 7 more 13's. This is, of course, equivalent to adding 13 more 7's, so our second-least value of n must be 10 + 13 = 23, and indeed $7 \times 23 = 161$. Finally, the sum of these two solutions is 10 + 23 = **33**. Note: There is no call to find the modulo 13 value of this sum.

5. Each Cat in the Hat, when wearing his hat, is 4 parts cat and 1 part hat, which is 5 parts in all. We are told that the original Cat in the hat is 6 feet, or 72 inches tall, so his hat must be 72 ÷ 4 = 18 inches tall. Cat A must be $\frac{4}{5}$ of this height, which is $\frac{4}{5}\times 18 = \frac{72}{5}$ inches. Cat A's hat must be $\frac{72}{5}\div 4 = \frac{18}{5}$ inches. Cat B must be be $\frac{4}{5}$ of this height, which is $\frac{4}{5}\times\frac{18}{5} = \frac{72}{25}$ inches. Cat B's hat must be $\frac{72}{25}\div 4 = \frac{72}{100}$ inches. Finally, Cat in the hat C must be $\frac{4}{5}$ of this height, which is $\frac{4}{5}\times\frac{72}{100} = \frac{72}{125}$ inches. If we write this as a decimal, we get exactly **0.576 inches**.

6. Using the values for A through E in the expression given, we obtain the following result:

$$\sqrt{0.1A + B + 10C + 100D + 1000E}$$

$$\sqrt{0.1\times 360 + (-10) + 10\times 62.8 + 100\times 33 + 1000\times 0.576}$$

$$\sqrt{36 - 10 + 628 + 3300 + 576}$$

$$\sqrt{4530}$$

$$\approx \mathbf{67.3}$$