IMLEM Meet #3
January 2013

Intermediate Mathematics League of Eastern Massachusetts

Category 1
Mystery
Meet #3, January 2013

1. The words shown below are special in that they each use only one vowel repeatedly. For each word, find the fraction of letters that are vowels, then write the two fractions in order from least to greatest on the answer line.

ABRACADABRA EFFERVESCENCE

2. It's pouring rain outside and Jonathan has two leaks in his roof. The first leak drips at a rate of 1 liter every 45 minutes and the other at 1 liter every 30 minutes. Assuming Jonathan has big enough buckets, how many hours will it take for him to collect 10 liters of water from the two leaks?

3. If this spiral lattice continues in the same manner, what number will be directly below 400?

.	21	22	23	24	25	26
.	20	7	8	9	10	27
.	19	6	1	2	11	28
39	18	5	4	3	12	29
38	17	16	15	14	13	30
37	36	35	34	33	32	31

Answers

1. __________,__________

2. ______________ hours

3. _____________________

Solutions to Category 1
Mystery
Meet #3, January 2013

Answers

1. $\frac{5}{13}, \frac{5}{11}$
2. 3 hours
3. 483

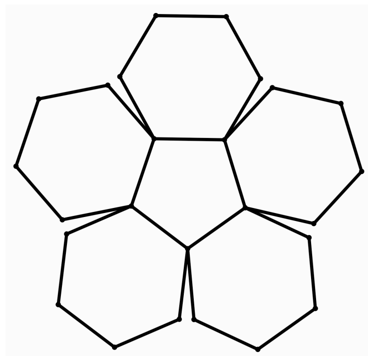
1. The word "ABRACADABRA" has 5 repeat vowels and 11 letters. "EFFERVESCENCE" has 5 repeat vowels and 13 letters. Thirteenths are smaller than elevenths and we have five of each, so the fractions, in order from least to greatest, are $\frac{5}{13}, \frac{5}{11}$.

2. Every 90 minutes, Jonathan will collect 2 liters from the first drip and 3 liters from the second drip. That's 5 liters. He will have 10 liters after 180 minutes or **3 hours**.

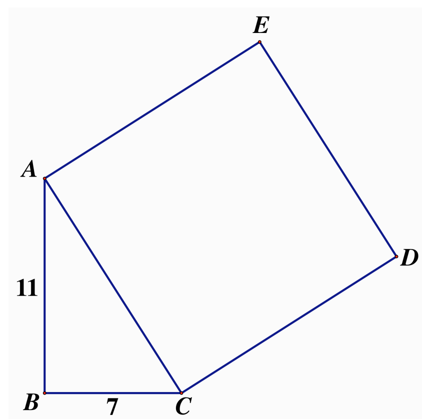
3. It helps to notice that the even square numbers are found on a diagonal that moves down and to the left. Directly below each square number is a number that is one less than the next even square number. Thus, under 400, which is 20^2, is $22^2 - 1 = 484 - 1 =$ **483**.

Category 2
Geometry
Meet #3, January 2013

1. Mia drew regular hexagons on each side of a pentagon. If she draws all the diagonals in all six shapes, how many diagonals will she have to draw? Note: A diagonal is a line segment that connects two vertices that are not already connected by a side.



2. Right triangle ABC below has legs of length 7 units and 11 units. How many square units are there in the area of square ACDE which is constructed on the hypotenuse of this triangle?



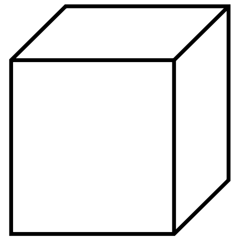
3. A small rectangular box has sides of lengths 6 cm, 6 cm, and 7 cm. How many centimeters are there in the space diagonal of the box? Note: A space diagonal is a line that connect two opposite vertices of the box and goes through the interior space of the box.

Answers

1. __________ diagonals

2. ___________ sq. units

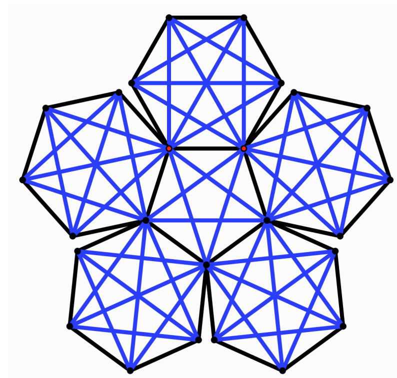
3. ________________ cm



Solutions to Category 2
Geometry
Meet #3, January 2013

Answers	
1.	50 diagonals
2.	170 sq. units
3.	11 cm

1. Three diagonals can be drawn from each of the six vertices on a hexagon, but this would count each diagonal at both ends. So there are 3 × 6 ÷ 2 = 9 diagonals in each hexagon. Similarly, there are 2 × 5 ÷ 2 = 5 diagonals in the pentagon. Mia will have to draw 5 × 9 + 5 = **50 diagonals**.



2. According to the Pythagorean theorem, the sum of the areas of the squares on the legs of a right triangle is equal to the area of the square on the hypotenuse. If we were to construct a square on leg AB, it would have an area of 11 × 11 = 121 square units. A square on leg BC would have an area of 7 × 7 = 49 square units. Their sum is 121 + 49 = **170 square units** and this is the area of square ACDE.

3. We can calculate the length of the space diagonal of the box by using the Pythagorean theorem twice. First we can find the length of the diagonal of the bottom face, which is $\sqrt{6^2+6^2}=\sqrt{72}=6\sqrt{2}$ cm. Then we find the length of the space diagonal using this length and the height of the box as follows: $\sqrt{7^2+\left(6\sqrt{2}\right)^2}=\sqrt{49+72}=\sqrt{121}=\mathbf{11}$ cm.

Alternatively, we can use a 3-dimensional version of the Pythagorean theorem as follows: $\sqrt{6^2+6^2+7^2}=\sqrt{36+36+49}=\sqrt{121}=\mathbf{11}$ cm.

Category 3
Number Theory
Meet #3, January 2013

1. Complete the base-six addition problem below, giving your answer in base six.

$$\begin{array}{r} 3452_{six} \\ +\ 2405_{six} \\ \hline \end{array}$$

2. Simplify the expression below. Write your answer in scientific notation.

$$\frac{7.2\times10^{-3}}{4.8\times10^{7}}\times\left(3\times10^{12}\right)$$

3. Find the base-six value of the following expression.

$$111_2+222_3+333_4+444_5$$

Answers

1. ____________ base six

2. ____________________

3. ____________ base six

Solutions to Category 3
Number Theory
Meet #3, January 2013

Answers

1. 10301 base six
2. 4.5×10^2
3. 1004 base six

1. When working in base 6, we have to remember to regroup whenever we get six in a place value. The number of groups of six will "carry" to the next place value, as shown below.

$$\begin{array}{r} 3452_{six} \\ +\ 2405_{six} \\ \hline 10301_{six} \end{array}$$

2. When evaluating this product, we can deal with the powers of ten separately. Also, we can "cancel" a common factor of 2.4 among the other factors, as shown below.

$$\frac{7.2\times10^{-3}}{4.8\times10^{7}}\times\left(3\times10^{12}\right)=\frac{7.2\times3}{4.8}\times\frac{10^{-3}\times10^{12}}{10^{7}}=\frac{3\times3}{2}\times10^{12-3-7}=4.5\times10^{2}$$

3. It probably makes sense to convert all these numbers to base ten first. Recall that the place values in any base are powers of the base. We will multiply the digits in each place value by the base-ten equivalents of the place values as follows.

$$111_2+222_3+333_4+444_5=$$
$$\left(1\times4+1\times2+1\times1\right)+\left(2\times9+2\times3+2\times1\right)+$$
$$\left(3\times16+3\times4+3\times1\right)+\left(4\times25+4\times5+4\times1\right)=$$
$$7+26+63+124=220_{ten}$$

Notice that 7, 26, 63, and 124 are all one less than the next power of the base. In other words, $\left(8-1\right)+\left(27-1\right)+\left(64-1\right)+\left(125-1\right)=220_{ten}$.

Finally, we convert 220_{ten} to base six as follows:

$$220_{ten}=216+4=1\times6^3+0\times6^2+0\times6^1+4\times6^0=\mathbf{1004_6}$$

Category 4
Arithmetic
Meet #3, January 2013

1. Evaluate the following expression. You should get a whole number.

$$\sqrt{3^2+3^3}+\sqrt{8^2+8^3}$$

2. Evaluate the expression below for a = 18 and b = 24. Express your answer as a mixed number in lowest terms.

$$\left(\frac{a^3}{b^2}\right)^0+\left(\frac{a^{-3}}{b^{-2}}\right)^{-1}$$

3. How many whole numbers are there between $\sqrt[4]{1000}$ and $\sqrt{1000}$?

Answers

1. ____________________

2. ____________________

3. _______ whole numbers

Solutions to Category 4
Arithmetic
Meet #3, January 2013

Answers
1. 30
2. $11\frac{1}{8}$
3. 26 whole numbers

1.
$$\sqrt{3^2+3^3}+\sqrt{8^2+8^3}=$$
$$\sqrt{9+27}+\sqrt{64+512}=$$
$$\sqrt{36}+\sqrt{576}=6+24=30$$

2. Notice that the first term in this expression is raised to the zero power, so it's value is 1. As for the second term, we substitute 18 for *a* and 24 for *b*, and evaluate as follows:

$$\left(\frac{a^{-3}}{b^{-2}}\right)^{-1}=\left(\frac{18^{-3}}{24^{-2}}\right)^{-1}=\frac{24^{-2}}{18^{-3}}=\frac{18^3}{24^2}=\frac{18\times18\times18}{24\times24}=\frac{3\times3\times9}{4\times2}=\frac{81}{8}=10\frac{1}{8}$$

Putting the two terms together, we get $\left(\frac{a^3}{b^2}\right)^0+\left(\frac{a^{-3}}{b^{-2}}\right)^{-1}=1+10\frac{1}{8}=\mathbf{11\frac{1}{8}}$.

3. We need to find a perfect fourth power that is slightly less than 1000 and a perfect square that is slightly greater than 1000. The two short lists below can help. From the list on the left, we see that $\sqrt[4]{1000}$ must be greater than 5, but less than 6. From the list on the right, we see that $\sqrt{1000}$ must be greater than 31, but less than 32. There are 31 whole numbers less than 32, but we need to exclude 5 of these, so there are 31 – 5 = **26 whole numbers** between 5 and 32.

$4^4=256$	$30^2=900$
$5^4=625$	$31^2=961$
$6^4=1296$	$32^2=1024$

Category 5
Algebra
Meet #3, January 2013

1. Solve the following inequality. Write your solution with the *x* on the left, the appropriate inequality sign in the middle, and a mixed number in lowest terms on the right.

$$7(5x+19)-2>40$$

2. For his birthday, Jarod received $54 from his Aunt Edna and three gift cards from his Uncle Joe. Although the gift cards are for different stores, each gift card has the same dollar value. If the absolute value of the difference between the money from Aunt Edna and the value of the three gift cards from Uncle Joe is $18, what is the absolute value of the difference between the two possible amounts that each gift card is worth?

3. How many integer values of *n* make the following inequality true?

$$2<\left|\frac{20}{n}\right|$$

Answers

1. ____________________

2. $ __________________

3. ___________ integers

Solutions to Category 5
Algebra
Meet #3, January 2013

Answers
1. $x > -2\frac{3}{5}$
2. \$12
3. 18 integers

1.

$$\begin{aligned} 7(5x+19)-2 &> 40 \\ 7(5x+19) &> 42 \\ 5x+19 &> 6 \\ 5x &> -13 \\ x &> \frac{-13}{5} \\ \boldsymbol{x} &> \boldsymbol{-2\frac{3}{5}} \end{aligned}$$

or

$$\begin{aligned} 7(5x+19)-2 &> 40 \\ 35x+133-2 &> 40 \\ 35x+131 &> 40 \\ 35x &> -91 \\ 5x &> -13 \\ \boldsymbol{x} &> \boldsymbol{-2\frac{3}{5}} \end{aligned}$$

2. We can capture this idea in the absolute value equation $|54-3x|=18$, where *x* is the unknown value of each gift card. To solve this, we consider two separate equations as shown above. The absolute value of the difference between \$24 and \$12 is \$24 – \$12 = **\$12**.

$$\begin{aligned} 54-3x &= 18 \\ 54-18 &= 3x \\ 36 &= 3x \\ x &= 12 \end{aligned} \qquad \begin{aligned} 54-3x &= -18 \\ 54+18 &= 3x \\ 72 &= 3x \\ x &= 24 \end{aligned}$$

3. Since there is an absolute value sign in this inequality, we need to solve two separate inequalities as shown below.

$$\begin{aligned} 2 &< \frac{20}{n} \\ 2n &< 20 \\ n &< 10 \end{aligned} \qquad \textbf{\textit{and}} \qquad \begin{aligned} 2 &> -\frac{20}{n} \\ 2n &> -20 \\ n &> -10 \end{aligned}$$

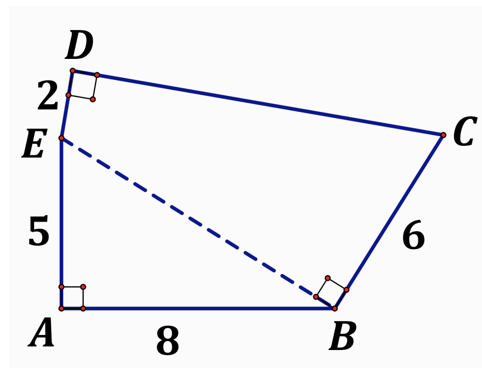
We need to find values of *n* that satisfy both inequalities, so it helps to write the solution as the compound inequality $-10 < n < 10$. The **18 integers** in this range are *n* = -9, -8, -7, -6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6, 7, 8, and 9. We have to skip *n* = 0, since we cannot divide by zero.

Category 6
Team Round
Meet #3, January 2013

1. The product of 98 and K is a perfect cube. What is the least possible positive integer value of K?

2. Solve the base-six equation $4(3x-34)+244=1000$ for x. Write your result in base six. Remember that all numbers shown in the equation are in base six.

3. Pentagon ABCDE has right angles EAB, CDE, and CBE. The lengths of sides DE, EA, AB, and CB are 2, 5, 8, and 6 cm, as shown. How many centimeters long is CD?



4. Evaluate

$$\sqrt{8\frac{1}{36}}+\sqrt[3]{-10\frac{81}{125}}+\sqrt{\frac{1}{9}+\frac{1}{16}}-\sqrt[3]{\frac{1}{8000}}.$$

5. Let A be the sum of the absolute values of the numbers in the set {–8, 11, k}, where k is an unknown integer. Let B be the absolute value of the sum of the numbers in the same set. List all possible positive differences between A and B.

6. Evaluate the expression below, using the answers the team obtained for questions 1, 3, and 4. For question 2, convert back to the *base-ten value* of your base-six answer. For question 5, use the *arithmetic mean* (average) of the numbers in your list.

Answers

1. ________________ = A
2. __________ base six ≠ B
3. _____________ cm = C
4. ________________ = D
5. ________________ ≠ E
6. ___________________

$$\frac{\sqrt{(A+C+D)(B+E)+4D}}{DE}$$

Solutions to Category 6
Team
Meet #3, January 2013

Answers
1. 28
2. 25 base six
3. 11 cm
4. 1
5. 16, 18, 20, 22
6. 2

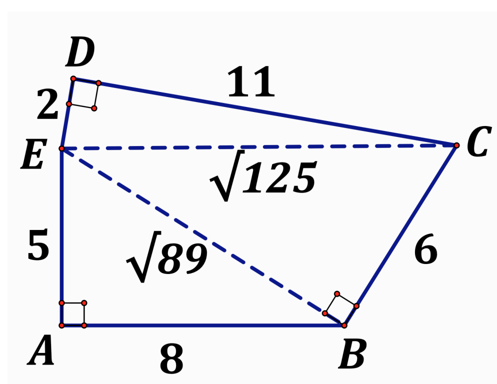
1. The prime factorization of 98 is 2 × 7 × 7. If the product 98*K* is to be a perfect cube, it will have to have three (or a multiple of three) of each prime factor. The least possible positive value of *K* to accomplish this is *K* = 2 × 2 × 7 = **28**. The product 98*K* would then be $2^3 \times 7^3 = 2744$, which is 14^3.

2. The equation is solved directly in base six on the left and converted to base ten and back on the right.

$$4(3x-34)+244=1000_{base\ six}$$

Base six	Base ten
$4(3x-34)+244=1000$	$4(3x-22)+100=216_{base\ ten}$
$4(3x-34)=312$	$4(3x-22)=116_{base\ ten}$
$3x-34=45$	$3x-22=29_{base\ ten}$
$3x=123$	$3x=51_{base\ ten}$
$x=\mathbf{25}$	$x=17_{base\ ten}$
	$x=\mathbf{25}_{base\ six}$

3. Using the Pythagorean theorem, we find that the length of EB is $\sqrt{5^2+8^2}=\sqrt{25+64}=\sqrt{89}$ cm. Similarly, the length of EC is $\sqrt{\left(\sqrt{89}\right)^2+6^2}=\sqrt{89+36}=\sqrt{125}$ cm. Finally, the length of DC is $\sqrt{\left(\sqrt{125}\right)^2-2^2}=\sqrt{125-4}=\sqrt{121}=$ **11 cm.**



4.

$$\sqrt{8\frac{1}{36}}+\sqrt[3]{-10\frac{81}{125}}+\sqrt{\frac{1}{9}+\frac{1}{16}}-\sqrt[3]{\frac{1}{8000}}=\sqrt{\frac{289}{36}}+\sqrt[3]{\frac{-1331}{125}}+\sqrt{\frac{25}{144}}-\sqrt[3]{\frac{1}{8000}}$$

$$\frac{17}{6}+\frac{-11}{5}+\frac{5}{12}-\frac{1}{20}=\frac{170}{60}+\frac{-132}{60}+\frac{25}{60}-\frac{3}{60}=\frac{60}{60}=\mathbf{1}$$

5. We need to try some different values of k and see what's going on. If $k = 0$, then $A=|-8|+|11|+|0|=8+11+0=19$, and $B=|-8+11+0|=|3|=3$. In this case, the positive difference between A and B is 19 – 3 = 16. If k has a positive value, such as $k = 1$, then $A=|-8|+|11|+|1|=8+11+1=20$, $B=|-8+11+1|=|4|=4$, and $A - B$ = 20 – 4 = 16. If $k = 2$, then we get $A - B$ = 21 – 5 is still 16. If k has a negative value, such as $k = -1$, then $A=|-8|+|11|+|-1|=8+11+1=20$, and $B=|-8+11+-1|=|2|=2$. This time $A - B$ = 20 – 2 = 18, which is a new difference. If $k = -2$, then we get $A - B$ = 21 – 1 = 20. If $k = -3$, then we get $A - B$ = 22 – 0 = 22. When $k = -4$, we get $A - B$ = 23 – 1 = 22, which is not a new difference. The possible positive differences are thus **16, 18, 20,** and **22**.

6. For question 2, we will use 17, since this is the base-ten value of $25_{\text{base six}}$. For question 4, we will use 19, which is the average of 16, 18, 20, and 22.

$$\frac{\sqrt{(A+C+D)(B+E)+4D}}{DE}=\frac{\sqrt{(28+11+1)(17+19)+4\times1}}{1\times19}$$

$$=\frac{\sqrt{40\times36+4}}{19}=\frac{\sqrt{1440+4}}{19}=\frac{\sqrt{1444}}{19}=\frac{38}{19}=\mathbf{2}$$