IMLEM Meet #2
November/December 2012

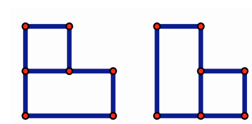
Intermediate Mathematics League of Eastern Massachusetts

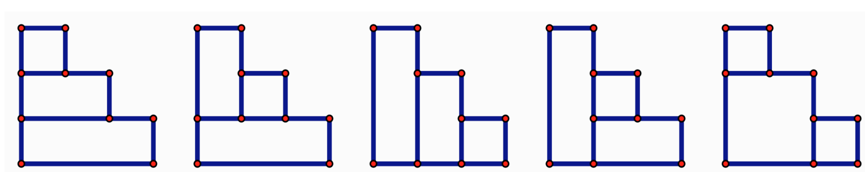
Category 1
Mystery
Meet #2, November/December 2012

1. Leo was supposed to divide a number by $3\frac{1}{2}$, but he multiplied the number by $3\frac{1}{2}$ instead and got 98. What answer should Leo have gotten?

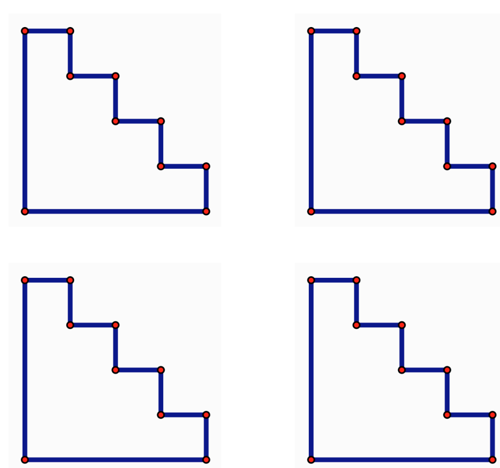
2. Working alone, Grandpa Greg can stack a cord of wood in 2 hours. When his granddaughter Gretchen works with him, they can stack a cord of wood in 56 minutes. How many minutes would it take Gretchen to stack a cord of wood by herself?

3. A 2-step diagram can be covered by 2 rectangles in 2 ways, as shown at right. A 3-step diagram can be covered by 3 rectangles in 5 ways, as shown below. How many ways can a 4-step diagram be covered by 4 rectangles?





You can use the templates below to help you draw 4-step diagrams.



Answers

1. ____________________

2. ___________ minutes

3. ______________ ways

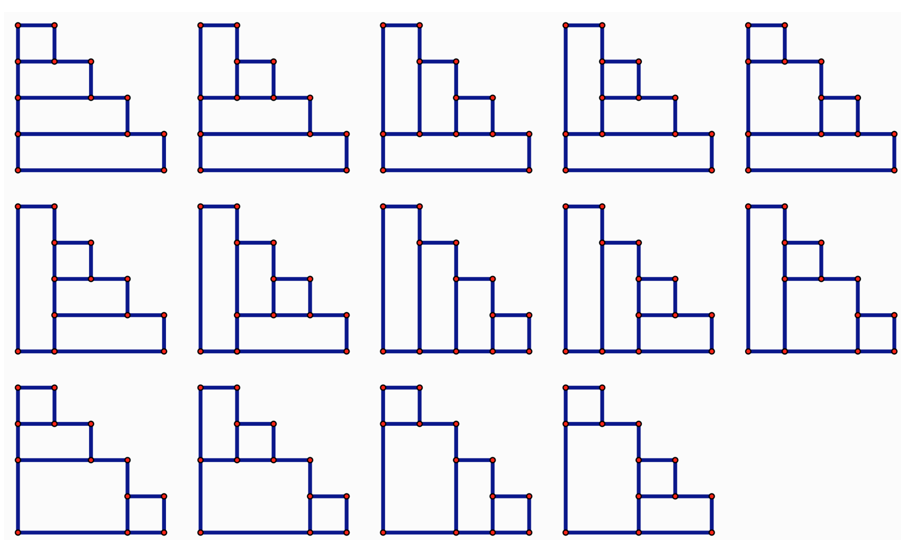
Solutions to Category 1
Mystery
Meet #2, November/December 2012

Answers
1. **8**
2. **105 minutes**
3. **14 ways**

1. We need to divide 98 by $3\frac{1}{2} = \frac{7}{2}$ twice. Dividing by $\frac{7}{2}$ is the same as multiplying by $\frac{2}{7}$, so the starting number must have been $98 \times \frac{2}{7} = 14 \times 2 = 28$, and the answer Leo should have gotten is $28 \times \frac{2}{7} = 4 \times 2 = \mathbf{8}$.

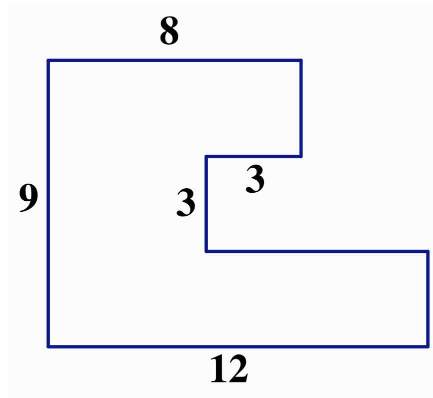
2. Since Gretchen and her grandfather took 56 minutes to stack the cord of wood, they must have stacked 1/56th of a cord every minute. Since Grandpa Greg takes 2 hours or 120 minutes to stack a cord of wood, he must stack 1/120th of a cord every minute. Gretchen's rate of stacking wood is the difference 1/56 – 1/120, which is 15/840 – 7/840 = 8/840 = 1/105th of a cord every minute. Thus it would take Gretchen **105 minutes** to stack a cord of wood by herself.

3. A 4-step diagram can be covered by 4 rectangles in the **14 ways** shown below. Notice that the first row shows the five 3-step diagrams with a new 1-by-4 block underneath and the second row shows the five 3-step diagrams with a new 4-by-1 block at the left. The third row uses a new 2-by-3 block with the two possible 2-step diagrams.



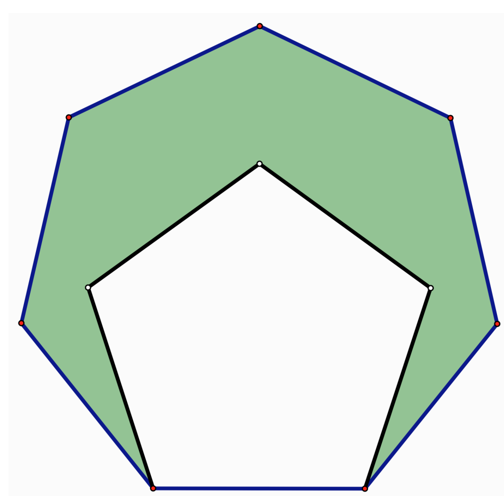
Category 2
Geometry
Meet #2, November/December 2012

1. Find the number of inches in the perimeter of the figure at right. All angles are right angles and all lengths are in inches.



2. Five squares of gold all have the same thickness, but they have edge lengths of 1 cm, 5 cm, 7 cm, 7 cm, and 11 cm. If the gold is melted down and recast with the same thickness as before into five identical squares, how many centimeters are there in the edge length of each square?

3. A formula for the area of a regular polygon is $A = 0.5ap$, where a is the length of the apothem and p is the perimeter. The apothem of a regular polygon is a segment that joins the polygon's center to the midpoint of any side. If the apothem of the pentagon below is about 0.69 units, the apothem of the heptagon is about 1.04 units, and their shared edge is 1 unit, find the number of square units in the area of the shaded region between the pentagon and the heptagon. Round your answer to the nearest tenth after your calculations are complete.

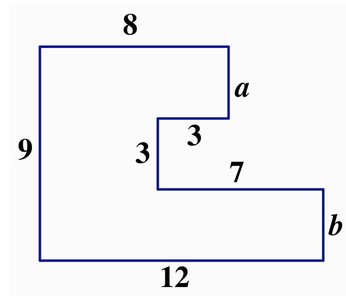


Answers
1. ____________ inches
2. ________________ cm
3. ___________ sq. units

Solutions to Category 2
Geometry
Meet #2, November/December 2012

Answers
1. **48 inches**
2. **7 cm**
3. **1.9 square units**

1. The total vertical rise on the right side of the figure must be equal to the 9 inches we see on the left side of the figure, so $a + 3 + b = 9$. Similarly, the total of the horizontal lengths would equal the 12 inches we see on the bottom, but there is an extra 3 + 3 = 6 inches because the figure turns into itself for 3 inches and then must come back 3 inches. The total perimeter is thus 9 + 12 + 9 + 12 + 6 = **48** inches.



2. The total surface area of the 5 squares is $1^2 + 5^2 + 7^2 + 7^2 + 11^2 =$ $1 + 25 + 49 + 49 + 121 = 245$ square cm. If the gold is to be recast in 5 equal squares of the same thickness as before, they must each have a surface area of 245 ÷ 5 = 49 square cm. The side length of each square would be $\sqrt{49}$ = **7 cm**. Notice that we squared the side lengths, then averaged these squared numbers, and finally took the square root of this average. This square root of the mean of the squares is called the "root mean square" or the "quadratic mean."

3. The area of the heptagon is 0.5 × 7 × 1.04 = 3.64 square units. The area of the pentagon is 0.5 × 5 × 0.69 = 1.725. The difference is 3.64 – 1.725 = 1.915, which is **1.9 square units** to the nearest tenth.

Category 3
Number Theory
Meet #2, November/December 2012

1. Some IMLEM Clusters will take Meet #2 on 11/29, which we will call a prime date, since 1129 is a prime number. Other Clusters will take Meet #2 on 12/06, which we will call a composite date. Give the prime factorization of 1206. You can use exponents or not, but you must write the prime factors in order from least to greatest.

2. Grace was supposed to find the greatest common factor (GCF) of 330 and 462, but she found the greatest common prime factor (GCPF) instead. How many times greater is the actual GCF of 330 and 462 than the GCPF that Grace found?

3. At the Gadgets and Gizmos factory, workers complete a gadget every 588 minutes and a gizmo every 882 minutes. The factory is open 24 hours a day, seven days a week, and the workers always complete the gadgets and gizmos on schedule. If it happens that a gadget and a gizmo are completed at the same time at 2:00 PM on a Thursday, how many additional times in the next week will a gadget and a gizmo be completed at the same time?

Answers

1. ____________________

2. _____________ times

3. _____________ times

Solutions to Category 3
Number Theory
Meet #2, November/December 2012

Answers
1. **$2 \times 3^2 \times 67$**
or **$2 \times 3 \times 3 \times 67$**
2. **6 times**
3. **5 times**

1. A systematic way to find the prime factorization of a number is to divide out primes in order of the primes as shown in the so-called ladder at right. The prime factorization of 1206 is **$2 \times 3 \times 3 \times 67$** or, using exponents, it's **$2 \times 3^2 \times 67$**.

$$\begin{array}{r|l} 2 & \underline{1206} \\ 3 & \underline{603} \\ 3 & \underline{201} \\ 67 & \underline{67} \\ & 1 \end{array}$$

2. The prime factorization of 330 is $2 \times 3 \times 5 \times 11$ and that of 462 is $2 \times 3 \times 7 \times 11$. The GCPF is 11 and the GCF is $2 \times 3 \times 11 = 66$, so the actual GCF is **6 times** greater than the GCPF.

3. The prime factorization of 588 is $2^2 \times 3 \times 7^2$, and that of 882 is $2 \times 3^2 \times 7^2$. The LCM of these two numbers is $2^2 \times 3^2 \times 7^2 = 1764$. This means that a gadget and a gizmo are completed simultaneously every 1764 minutes. There are $7 \times 24 \times 60 = 10{,}080$ minutes in a week. Five times 1764 = 8820 and $6 \times 1764 = 10584$, so gadgets and gizmos will be completed simultaneously an additional **5 times** in the next week.

Category 4
Arithmetic
Meet #2, November/December 2012

1. What number is 20% less than $9\frac{2}{7}$?
Express your answer as a mixed number in lowest terms.

2. Simplify the expression below to a common fraction.

$$\frac{\frac{3}{11}+0.\overline{24}}{\frac{4}{9}+0.\overline{41}}$$

3. What is the 202nd digit to the right of the decimal point in the decimal equivalent of $\frac{25}{202}$?

Answers

1. ______________

2. ______________

3. ______________

Solutions to Category 4
Arithmetic
Meet #2, November/December 2012

Answers
1. $\mathbf{7\frac{3}{7}}$
2. $\mathbf{\frac{3}{5}}$
3. **2**

1. Twenty percent less than all of a number is 80% or 4/5 *of* that number. Multiplying 4/5 by $9\frac{2}{7}$, we get

$\frac{4}{5} \times 9\frac{2}{7} = \frac{4}{5} \times \frac{65}{7} = \frac{4 \times 13}{7} = \frac{52}{7} = \mathbf{7\frac{3}{7}}$.

2. We can simplify the expression as follows:

$$\frac{\frac{3}{11} + 0.\overline{24}}{\frac{4}{9} + 0.\overline{41}} = \frac{\frac{3}{11} + \frac{24}{99}}{\frac{4}{9} + \frac{41}{99}} = \frac{\frac{27+24}{99}}{\frac{44+41}{99}} = \frac{\frac{51}{99}}{\frac{85}{99}} = \frac{51}{85} = \mathbf{\frac{3}{5}}$$

3. We have to do long division to find the repeating decimal pattern for 25/202, as shown at right. The first digit to the right of the decimal point is not part of the repeating pattern. Then we get a repeating pattern of four digits. We know it repeats because we get a repeat remainder of 48. The 202nd digit to the right of the decimal point is the 201st digit in the pattern. Since 201 is 1 more than a multiple of 4, we get the first digit in the pattern, which is **2**.

$$\begin{array}{r}
0.1\overline{2376} \\
202\,\big)\,25.00000 \\
-202 \\
480 \\
-404 \\
760 \\
-606 \\
1540 \\
-1414 \\
1260 \\
-1212 \\
48
\end{array}$$

Category 5
Algebra
Meet #2, November/December 2012

1. Five consecutive multiples of 17 have a sum of 510. What is the greatest of these five multiples of 17?

2. During the summer of 2012, Joel earned $27 less than three times as much as he earned during the summer of 2011. For the summer of 2013, he hopes to earn $27 less than three times his earnings of the summer of 2012. If all goes according to his plan and we know that he earned $714 during the summer of 2012, how many more dollars will he earn in the summer of 2013 than he did in the summer of 2011?

3. The formula $S = \frac{n \times (a + l)}{2}$ can be used to find the sum *S* of an arithmetic sequence, where *n* is the number of terms, *a* is the first term, and *l* is the last term. The last term can be computed using the formula $l = a + (n - 1) \times d$, where *a* and *n* are the same as above, and *d* is the common difference between terms. What is the common difference *d* if the sum of 17 terms is 1479 and the first term is 39?

Answers
1. ____________
2. $ ____________
3. ____________

Solutions to Category 5
Algebra
Meet #2, November/December 2012

Answers
1. **136**
2. **$1868**
3. **6**

1. The sum of the five multiples of 17 is five times the value of the middle number. Dividing 510 by 5, we find that the middle number must be 102, which is 6 × 17. The greatest of the five multiples of 17 must be 102 + 17 + 17 = **136**, which is 8 × 17**.**

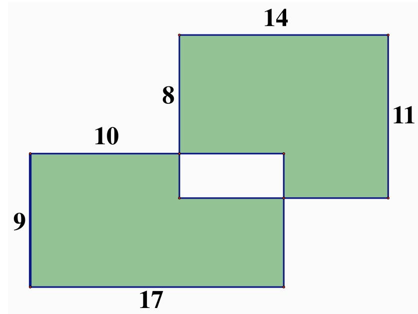
2. During the summer of 2011, Joel must have earned (714 +27) ÷ 3 = 741 ÷ 3 = $247. In the summer of 2013, he hopes to earn 3 × 714 – 27 = 2142 – 27 = $2115. The difference is 2115 – 247 = **$1868**.

3. Substituting the known values into the first equation, we get the equation $1479 = \frac{17 \times (39 + l)}{2}$, which we can solve for *l*. We multiply both sides by 2 and get $2958 = 17 \times (39 + l)$. Dividing both sides by 17, we get $174 = 39 + l$. This means the last term must be 174 – 39 = 135. Now we substitute the known values into to the second equation and solve for *d*: $135 = 39 + (17 - 1) \times d$. This simplifies to $96 = 16d$, so *d* = **6.**

Category 6
Team Round
Meet #2, November/December 2012

1. The product of two natural numbers *a* and *b* is 770. What is the GCF of *a* and *b*?

2. What's the sum of all the natural numbers less than 100 that have three distinct primes in their prime factorization?



3. Two rectangles overlap at right angles as shown above. The lengths given are all in centimeters. How many square centimeters are there in the sum of the shaded regions?

4. In a counting contest, player A begins counting down from 6265 by 17s and player B begins counting up from 2385 by 23s. If they begin at the same time and they count at the same rate, what number will they say at the same time?

5. Starting with $n = 2$ and allowing n to increase through natural numbers, consider the unit fractions $1/n$. If $1/n$ has a terminating decimal equivalent, place that decimal equivalent in list A. If $1/n$ has a repeating decimal equivalent, place that decimal equivalent in list B, using bar notation at the earliest opportunity. Let T be the total number of digits to the right of the decimal point for the first eight numbers on list A, and let R be the total number of digits to the right of the decimal point for the first eight numbers on list B. Find the positive difference between R and T.

6. Evaluate the expression below, using the values the team obtained in questions 1, 2, 3, and 5. For question 4, use D_p, which is the largest prime factor of the answer to question 4.

$$\frac{\sqrt{B+C-A}-\sqrt{D_p-A}}{E+A}$$

Answers

1. ______________ = A
2. ______________ = B
3. ________ sq. cm = C
4. ______________ = D
5. ______________ = E
6. ________________

Solutions to Category 6
Team
Meet #2, November/December 2012

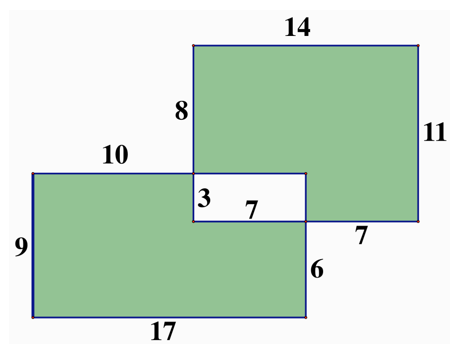
Answers	
1.	**1**
2.	**520**
3.	**265 sq. cm**
4.	**4616**
5.	**12**
6.	**280**

1. The prime factorization of 770 is $2 \times 5 \times 7 \times 11$. Since there is only one of each prime factor in this product, there is no way that the numbers *a* and *b* can have a common prime factor. This guarantees that the GCF of *a* and *b* is **1**. Some teams will verify that the GCF is indeed 1 for the following eight possible pairs of numbers: 1×770, 2×385, 5×154, 7×110, 11×70, 10×77, 14×55, and 22×35.

$2 \times 3 \times 5 = 30$
$2^2 \times 3 \times 5 = 60$
$2 \times 3^2 \times 5 = 90$
$2 \times 3 \times 7 = 42$
$2^2 \times 3 \times 7 = 84$
$2 \times 3 \times 11 = 66$
$2 \times 3 \times 13 = 78$
$2 \times 5 \times 7 = 70$

2. The numbers shown at right have three distinct primes in their prime factorization. Their sum is **520**.

3. If we calculate the area of each rectangle and add them, we get $9 \times 17 = 153$ and $11 \times 14 = 154$, for a total of 307 square cm. When the two rectangles are overlapped, an area of $3 \times 7 = 21$ must be subtracted *twice*. The remaining area is $307 - 42 =$ **265 square cm**.



4. Player A's number pattern is $6265 - 17x$ and player B's number pattern is $2385 + 23x$. When they say the same number, these two expressions will be equal, so we need to solve the equation $6265 - 17x = 2385 + 23x$. Adding $17x$ to both sides of the equation and subtracting 2385 from both sides of the equation, we get $3880 = 40x$. Dividing 3880 by 40, we get $x = 97$. This means they will each say 97 numbers. The number they both say is $6265 - 17 \times 97 = 6265 - 1649 =$ **4616**. To verify this, we can calculate $2385 + 23 \times 97 = 2385 + 2231 = 4616$.

5. List A and list B are shown below. List A has a total of 16 digits to the right of the decimal point and list B has a total of 28 digits to the right of the decimal point, so the desired difference is $R - T = 28 - 16 =$ **12**.

List A	List B
$\frac{1}{2} = 0.5$	$\frac{1}{3} = 0.\overline{3}$
$\frac{1}{4} = 0.25$	$\frac{1}{6} = 0.1\overline{6}$
$\frac{1}{5} = 0.2$	$\frac{1}{7} = 0.\overline{142857}$
$\frac{1}{8} = 0.125$	$\frac{1}{9} = 0.\overline{1}$
$\frac{1}{10} = 0.1$	$\frac{1}{11} = 0.\overline{09}$
$\frac{1}{16} = 0.0625$	$\frac{1}{12} = 0.08\overline{3}$
$\frac{1}{20} = 0.05$	$\frac{1}{13} = 0.\overline{076923}$
$\frac{1}{25} = 0.04$	$\frac{1}{14} = 0.0\overline{714285}$

6. First we need to find the largest prime factor of 4616. The prime factorization of 4616 is $2^3 \times 577$, so $D_p = 577$. Substituting the values for A, B, C, D_p, and E into the expression and simplifying, we get:

$$\frac{\sqrt{B + C - A} - \sqrt{D_p - A}}{E + A}$$

$$\frac{\sqrt{520 + 265 - 1} - \sqrt{577 - 1}}{12 + 1}$$

$$\frac{\sqrt{784} + \sqrt{576}}{13}$$

$$\frac{28 + 24}{13} = \frac{52}{13} = \mathbf{4}$$