IMLEM Meet #1
October 2012

Intermediate Mathematics League of Eastern Massachusetts

Category 1
Mystery
Meet #1, October 2012

1. Each letter in the boxes below represents a number. The sum of the numbers in any three consecutive boxes is 13. What is the value of G?

A	5	C	D	E	F	G	H	6	J

2. Todd multiplied a natural number by 7 and got an answer whose every digit was a 3. What is the least possible natural number that Todd could have multiplied by 7?

$$7 \times \_\_\_\_\_ = 333\ldots3$$

3. There are 95 students at Melmi Middle School, where the ratio of girls to boys is 11 to 8. At the nearby Lemmi High School, there are 130 students and the ratio of girls to boys is 8 to 5. If the schools were combined into one, what would be the ratio of girls to boys? Express your answer as a reduced ratio of whole numbers in the form a:b.

Answers

1. _____
2. _____
3. _____:_____

Solutions to Category 1
Mystery
Meet #1, October 2012

Answers
1. **2**
2. **47,619**
3. **3:2**

1. Since both A + 5 + C = 13 and 5 + C + D = 13, we must have A = D. If we continue this reasoning, we find that E and H have to be 5. Similarly, working from right to left, C and F have to be 6. For the sum of the numbers in any three consecutive boxes to be 13, the other letters, including G, must all be **2**.

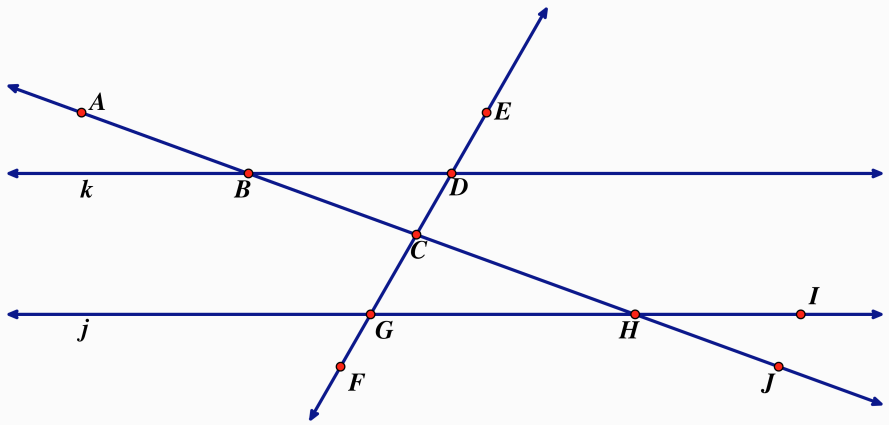
2. The units digit of the unknown number must be a 9, since 7 × 9 = 63. This results in a 6 being carried to the next place value. Now we want to multiply by 1, since 7 × 1 + 6 = 13 will give us another 3. We can continue building the number from right to left in this manner. The result is shown at right.

$$\begin{array}{r} 47619 \\ \times \quad 7 \\ \hline 333333 \end{array}$$

3. At Melmi Middle School, the 11 parts girls and 8 parts boys is 19 parts in all, so there must be 95 ÷ 19 = 5 students in each part. That's 11 × 5 = 55 girls and 8 × 5 = 40 boys. At Lemmi High School, the 8 parts girls and 5 parts boys is 13 parts in all, so there must be 130 ÷ 13 = 10 students in each part. That's 8 × 10 = 80 girls and 5 × 10 = 50 boys. If the schools were combined, there would be 55 + 80 = 135 girls and 40 + 50 = 90 boys. These numbers are both multiples of 45, so we get a girl to boy ratio of 135:90 = **3:2**. (Note that Melmi is IMLEM backwards and Lemmi is an anagram of IMLEM.)

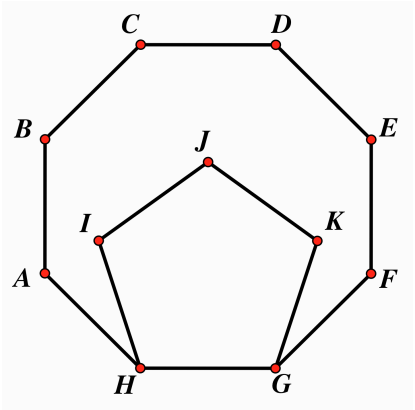
Category 2
Geometry
Meet #1, October 2012

1. In the diagram below, lines *k* and *j* are parallel. The measure of angle IHJ is 20 degrees and the measure of angle DCH is 80 degrees. How many degrees are there in the measure of angle CGH?



2. The supplement of a certain angle is 5 more than 6 times the angle itself. How many degrees are there in the measure of the complement of this angle?

3. In the figure at right, regular pentagon GHIJK is constructed inside regular octagon ABCDEFGH so that the two polygons share side GH. If line segments BC and IJ are extended so that they meet at a point X, how many degrees are there in the measure of angle BXI?



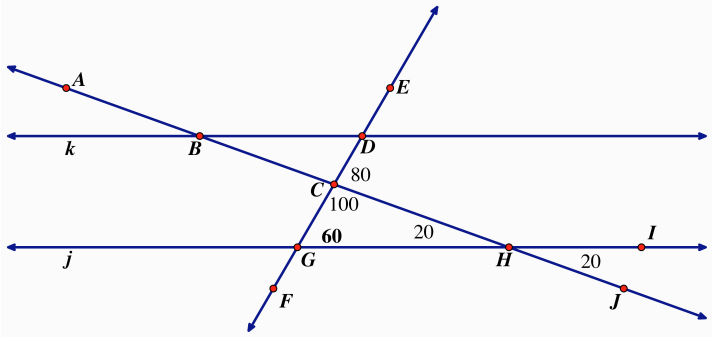
Answers

1. _____degrees
2. _____degrees
3. _____degrees

Solutions to Category 2
Geometry
Meet #1, October 2012

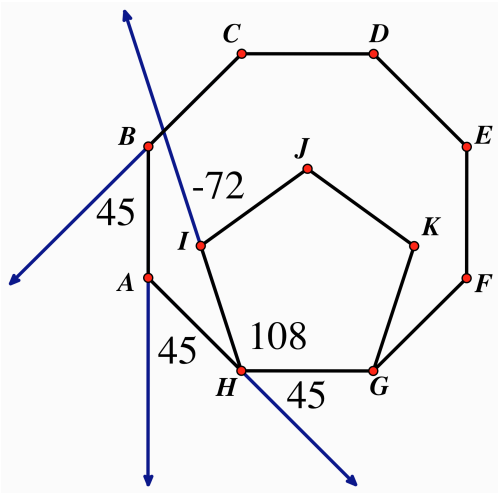
Answers
1. **60**
2. **65**
3. **9**

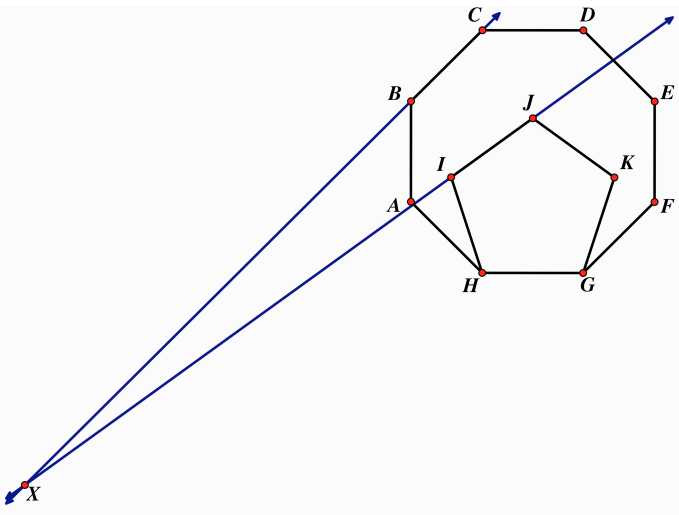
1. Angle CHG = 20 degrees (vertical angles). Angle GCH = 100 degrees (supplementary angles). Thus angle CGH = 180 – 100 – 20 = **60 degrees**.



2. Translating the words to algebra, we can write the equation $180 - x = 6x + 5$. This simplifies to $7x = 175$, so $x = 25$. The complement of x is therefore $90 - 25 =$ **65 degrees**.

3. Imagine that you are taking a walk along the path CBAHIJ. You will take two left turns of 45 degrees, one left turn of 45 + 108 = 153 degrees and one right turn of 72 degrees. The end result is that you have turned 45 + 45 + 153 – 72 = 171 degrees, which is 9 degrees less than 180 degrees. There must be **9 degrees** in the measure of angle BXI.





Category 3
Number Theory
Meet #1, October 2012

1. Two primes p and q have a sum of 38. Given that $p > q$, find the value of $p - q$.

2. What single-digit value of N will make the 7-digit number 1,295,N84 divisible by 18?

3. What number is the least three-digit multiple of 5 that has exactly six factors?

Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #1, October 2012

Answers
1. **24**
2. **7**
3. **175**

1. The two primes must be 31 and 7. Their positive difference is 31 – 7 = **24**.

2. The 7-digit number 1,295,*N*84 is clearly even, so we only need to make sure it is divisible by 9. The sum of the known digits is 1 + 2 + 9 + 5 + 8 + 4 = 29. *N* will have to be **7** to get us to the next multiple of 9.

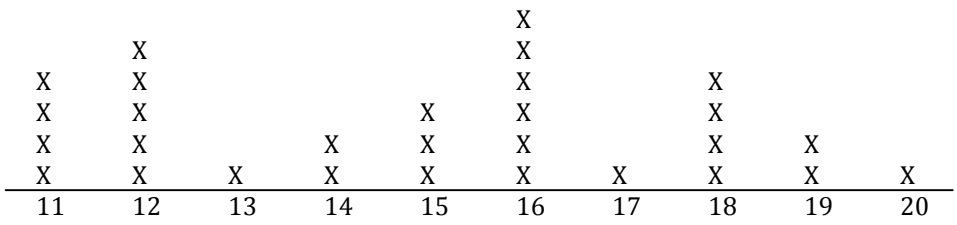
3. Numbers with exactly six factors must have a prime factorization of the form p^5 or $p^2 \times q$, where p and q are primes. The smallest multiple of 5 of the form p^5 is $5^5 = 3125$, which is not three digits. The other options are the forms $p^2 \times 5$ or $5^2 \times q$. The least three-digit number of the $p^2 \times 5$ form is $7^2 \times 5 = 49 \times 5 = 245$. The least three-digit number of the $5^2 \times q$ form is $5^2 \times 7 = 25 \times 7 =$ **175**, which is our desired answer.

Category 4
Arithmetic
Meet #1, October 2012

1. Evaluate the following expression according to the order of operations.

$$\frac{2+5\times 2^5-2\times 5^2+5}{2\times 5-2+5}$$

2. The line plot below shows the number of song tracks on a collection of CD's. Each X represents one CD. What is the positive difference between the median and the mode of the data?



Song Tracks on CD's

3. All 13 students who took a 12-point quiz received a positive integer score. The median was 10, the mean was 10, and the mode was 9. What is the maximum number of students who could have received perfect scores?

Answers

1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #1, October 2012

Answers	
1.	**9**
2.	**1**
3.	**4**

1. The correct evaluation is shown below.

$$\frac{2+5\times 2^5-2\times 5^2+5}{2\times 5-2+5}=\frac{2+5\times 32-2\times 25+5}{10-2+5}=\frac{2+160-50+5}{13}=\frac{117}{13}=9$$

2. There are 29 X's on the line plot, so the median is the 15th number from the left, which is 15. The mode is 16, which can be seen as the tallest column of X's. The positive difference between these two is 16 – 15 = **1**.

3. Let's make a row of 13 blank spaces. We can place a 10 in the middle, since 10 is the median. We can start to place 12's at the far right of the line, but we must have one more 9 than 12's so that 9 is still the mode. Finally, since the mean is 10, we have to make sure that the sum of the numbers is 13 × 10 = 130. One possible set of numbers is shown below. The numbers that are circled could be changed, but the maximum number of perfect scores is **4**.

Category 5
Algebra
Meet #1, October 2012

1. Evaluate the expression below if $a = \frac{1}{2}$, $b = \frac{2}{3}$, and $c = \frac{3}{4}$.
Express your answer as a mixed number in lowest terms.

$$\frac{ab + ac + bc}{abc}$$

2. Find the value of *M* that makes the equation below an identity.

$$11(4x + 9) - 19x + M = 76x - 17(3x - 5)$$

3. On the second day of a three-day math challenge, Julie completed 6 more than twice as many problems as she completed on the first day. On the third day, she completed 1 less than two thirds as many problems as she completed on the second day. If Julie completed 74 problems in all, how many problems did she complete on the first day?

Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #1, October 2012

Answers	
1.	$\mathbf{4\frac{5}{6}}$
2.	**–14**
3.	**15**

1. The correct evaluation is shown below.

$$\frac{\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{3}{4}+\frac{2}{3}\times\frac{3}{4}}{\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}}=\frac{\frac{1}{3}+\frac{3}{8}+\frac{1}{2}}{\frac{1}{4}}=\frac{\left(\frac{8}{24}+\frac{9}{24}+\frac{12}{24}\right)}{\frac{1}{4}}=\frac{4}{1}\times\frac{29}{24}=\frac{29}{6}=\mathbf{4\frac{5}{6}}$$

2. On the third line of the algebra below, we see $25x$ on each side. Since 25 times any real number is equal to 25 times the same number, the solution to the equation depends only on M. If M is **–14**, then x can be any real number, making the original equation an identity. If M is any other number, then there would be no solution to the original equation.

$$\begin{aligned}11(4x+9)-19x+M&=76x-17(3x-5)\\44x+99-19x+M&=76x-51x+85\\25x+99+M&=25x+85\\M&=85-99\\M&=\mathbf{-14}\end{aligned}$$

3. Let x be the number of problems Julie completed on the first day. Then we can write and solve the following equation.

$$\begin{aligned}x+(2x+6)+\left(\frac{2}{3}(2x+6)-1\right)&=74\\x+2x+6+\frac{4x}{3}+4-1&=74\\\frac{13x}{3}+9&=74\\\frac{13x}{3}&=65\\x&=\mathbf{15}\end{aligned}$$

Category 6
Team Round
Meet #1, October 2012

1. What is the greatest positive difference between consecutive primes that are both less than 200?

2. Kathryn multiplied two whole numbers together and got the product 100,000. If neither of the numbers had any zeros, what is the sum of the two numbers?

3. To train for a bicycle race, Claude starts with a "warm-up," biking at 9 mph for 20 minutes, and ends with a "cool-down," biking at 6 mph for 15 minutes. In between the warm-up and cool-down, he rides at an average speed of 18 mph. If the entire ride takes 3 hours, what is his average speed?

4. What fraction of the positive factors of 6300 are even? Express your answer as a common fraction.

5. Two cars traveled together on a long trip. They both started with 10 gallons of gasoline. Car A gets 20 miles per gallon and car B gets 30 miles per gallon. How many miles had they traveled when car B had twice as much gasoline left in its tank as car A had left in its tank?

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression:

$$\frac{B}{77}\left(\frac{\sqrt{C\times D\times E}}{A-4}\right)$$

Answers

1. _____ = *A*
2. _____ = *B*
3. _____ = *C*
4. _____ = *D*
5. _____ = *E*
6. _____

Solutions to Category 6
Team Round
Meet #1, October 2012

Answers
1. **14**
2. **3157**
3. **16 mph**
4. **2/3**
5. **150 miles**
6. **164**

1. The greatest positive difference between consecutive primes less than 200 is 127 – 113 = **14**.

2. The prime factorization of 100,000 = $2^5 \times 5^5$. The two numbers must have been 2^5 =32 and 5^5 = 3125. The desired sum is 32 + 3125 = **3157**.

3. First we will convert 20 minutes to 1/3 of an hour and 15 minutes to 1/4 of an hour. The in-between part of the ride must have lasted 3 – 1/3 – 1/4 = 36/12 – 4/12 – 3/12 = 29/12 hours. Claude must have 9 × 1/3 = 3 miles on his warm-up, 6 × 1/4 = 3/2 miles on his cool down, and 18 × 29/12 = 87/2 miles in between. That's a total of 48 miles in 3 hours, which is an average speed of **16 mph**.

4. The prime factorization of 6300 is $2^2 \times 3^2 \times 5^2 \times 7$. Raising each exponent by 1 and multiplying, we calculate that 6300 has 3 × 3 × 3 × 2 = 54 factors. (Do you know why this works?) Excluding the factors of 2, we note that $3^2 \times 5^2 \times 7$ has 3 × 3 × 2 = 18 odd factors. The remaining 36 factors must be even, so 36/54 =**2/3** of the fractions of 6300 are even.

5. Dividing the number of miles a car travels by its rate of fuel consumption gives the number of gallons of gasoline consumed. Subtracting the number of gallons consumed from the 10 gallons they started with will give the number of gallons of gasoline remaining in the tank. With this reasoning and the variable *x* for the number of miles traveled, we arrive at the equation $10-\frac{x}{30}=2\left(10-\frac{x}{20}\right)$. This simplifies to $300-x=600-3x$, then 2*x* = 300, then *x* =150 miles. The cars must have traveled **150 miles** when car B had twice as much gasoline in its tank as car A had left in its tank.

6. Substituting the values for *A* through *E* into the expression and simplifying, we get:

$$\frac{B}{77}\left(\frac{\sqrt{C\times D\times E}}{A-4}\right)=\frac{3157}{77}\left(\frac{\sqrt{16\times\frac{2}{3}\times 150}}{14-4}\right)$$

$$=41\left(\frac{\sqrt{16\times 100}}{10}\right)$$

$$=41\times\frac{4\times 10}{10}$$

$$=41\times 4$$

$$=\mathbf{164}$$