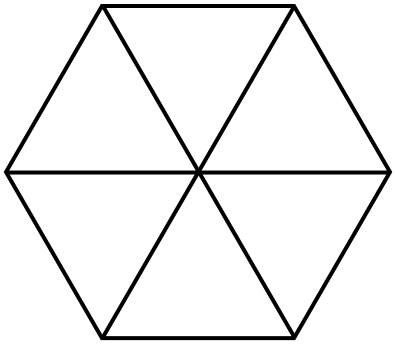
Intermediate Mathematics League of Eastern Massachusetts

Statistics and notes – not part of the original meet

Scheduled Meet Date	Jan. 12, 2012	
Number of Teams Competing	72	
Average Team Score	91	
Average Individual Score	6.3	(out of 18)

Category	1 Myst	2 Geom	3 NumTh	4 Arith	5 Alg
Number of Regulars Competing in This Category	424	410	412	420	416
Percent of Regulars with each possible score in the category:					
0	**42%**	30%	**57%**	26%	**39%**
2	39%	**33%**	22%	**32%**	24%
4	12%	24%	13%	24%	16%
6	7%	13%	8%	17%	21%

Category 1 – Mystery



1. The diagram shows a regular Hexagon, and the total length of lines shown in the diagram is 108 inches. How many inches are there in the sum of perimeters of all six triangles shown?

2. Bob has a third more money than Alice, but a third less than Cathy. Together, all three have $3,900. How much money does Cathy have?

3. Numbers are frequently represented in Base 16 in computers. In that base, the letters $A - F$ are the digits representing the values $10 - 15$. Solve the following problem and give your answer in Base 10:

$$(100 + B0)_{base\ 16} = ?_{base\ 10}$$

Answers

1. _____ inches
2. $ _____
3. _____

Solutions to Category 1 – Mystery

Answers
1. 162
2. $1,800
3. 432

1. The diagram includes lines that total 12 side's lengths, so each side measures $\frac{108}{12} = 9$ inches.

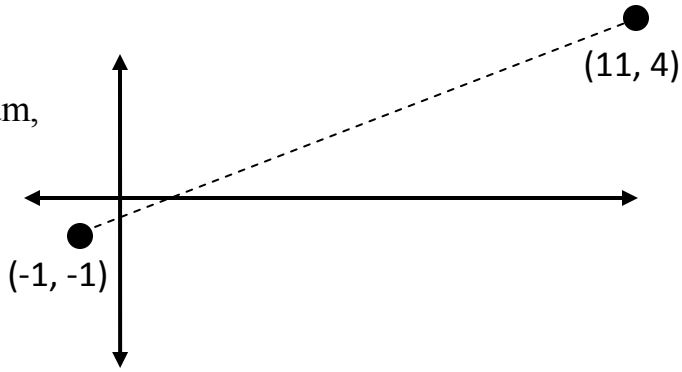
   Each triangle's perimeter then measures $3 \cdot 9 = 27$ inches, and all six together measure $6 \cdot 27 = 162$ inches.

2. If we call the amount that Bob has x, then Alice has $\frac{3}{4} \cdot x$ (adding a third of that will get us to x), and Cathy has $\frac{3}{2} \cdot x$ (subtracting a third of that will get us to x). The sum of their amounts is:
   $\frac{3}{4} \cdot x + x + \frac{3}{2} \cdot x = 3\frac{1}{4} \cdot x = \$3{,}900$ and so $x = \$1{,}200$ and Cathy's amount is $\frac{3}{2} \cdot x = \$1{,}800$

3. $(100 + B0)_{base\ 16} = 1B0_{base\ 16} = 1 \cdot 16^2 + 11 \cdot 16 + 0 = 256 + 176 = 432$

Category 2 – Geometry

1. Given the coordinates in the diagram, what is the distance between the two points?



2. How many diagonals are there in a regular polygon with 16 sides (a Hexadecagon)?

3. The sum of interior angles in a regular polygon is 24 times as great as the measure of each of its exterior angles.
   How many sides does the polygon have?

Answers
1. _____ Units
2. _____ Diagonals
3. _____ Sides

Solutions to Category 2 – Geometery

Answers
1. 13
2. 104
3. 8

1. The horizontal distance is 12 units, and the vertical distance is 5 units, so the total distance is $\sqrt{12^2 + 5^2} = 13$ units.

2. The formula for the number of diagonals in a polygon with N sides is: $\frac{N \cdot (N-3)}{2}$ so in our case we'll have $\frac{16 \cdot 13}{2} = 104$ diagonals.

3. The exterior angles of a polygon all add up to 360 degrees, so if there are N sides to the polygon, then each exterior angle measures $\frac{360}{N}$ degrees.
   Every interior angle measures $\frac{N-2}{N} \cdot 180$ degrees, and their sum is therefore $(N-2) \cdot 180$ degrees. So in our case we're told that: $(N-2) \cdot 180 = 24 \cdot \frac{360}{N}$ whice we can rewrite as: $N \cdot (N-2) = 24 \cdot \frac{360}{180} = 48$.
   Though this is technically a quadratic equation, we know that N is a natural number and can easily find that $N = 8$ is a solution (an Octagon).
   [The other solution, $(N = -6)$*, is clearly not an answer to our problem]*.

Category 3 – Number Theory

1. Solve the following Binary (Base 2) problem. *Give your answer in base* 10.
   $1{,}001{,}100 + 100 * 101{,}101 =?$

2. If stretched out, a DNA molecule can measure 64,000 nano-meters (a nano-meter is 10^{-9} of a meter).
   The diameter of Earth is 12,800 kilometers (a kilometer is 1,000 meters).
   How many stretched-out DNA molecules can we fit in the diameter?
   Express your answer in scientific notation.

3. All the numbers in this problem are in base 7. Your answer should also be expressed in base 7.
   $LCM(60{,}51) =?$
   $LCM \equiv Least\ Common\ Multiple$

Answers
1. _____
2. _____
3. _____

Solutions to Category 3 – Number Theory

1. We can do this in 2 steps:

   First: $100 * 101{,}101 = 10{,}110{,}100$

   (simply adding two zeroes).

   Then: 1,001,100

   $+10{,}110{,}100$

   100,000,000

   This Binary number stands for $2^8 = 256$

2. Converting both measures to meters, the answer will be:

   $$\frac{12{,}800 * 10^3\ meters}{64{,}000 * 10^{-9}\ meters} = \frac{12.8 * 10^6}{6.4 * 10^{-5}} = \frac{12.8}{6.4} * 10^{11} = 2 * 10^{11}$$

   Note that $12.8 * 10^6$ *is not a valid scientific notation (*$1.28 * 10^7$ *is*) *but it's easier to see the division this way.*

3. It is easiest to translate the numbers to base 10, solve, and then translate the answer back to base 7:

   $60_{base\ 7} = 6 \cdot 7 = 42$

   $51_{base\ 7} = 5 \cdot 7 + 1 = 36$

   $LCM(42{,}36) = LCM(2 \cdot 3 \cdot 7, 2^3 \cdot 3^2) = 2^2 \cdot 3^2 \cdot 7 = 252$

   $252 = 5 \cdot 49 + 7 = 510_{base\ 7}$

   [Note the similarity to the following base 10 *equality:*

   $LCM(90{,}81) = 810$ *where we substituted the digits appropriately].*

Answers

1. 256
2. $2 * 10^{11}$ or $2.00 * 10^{11}$
3. 510

Category 4 – Arithmetic

1. Evaluate the following expression:

$$(4^2 - 3^2)^2 - \sqrt{2^5 + 4^3 + 5^2} - \sqrt{2^3 + 5^3 + 6^2}$$

2. N is a natural number such that: $16^3 < N^5 < 20^3$. Find N.

3. Evaluate the expression below:

$$\left(\sqrt[3]{\frac{8}{27}}\right)^{-2} * \left(\frac{1}{2}\right)^{-4} * \left(\sqrt[4]{81}\right)^{-1} - 2^{\sqrt{2^3+1^3}}$$

Answers
1. _____
2. _____
3. _____

Solutions to Category 4 - Arithmetic

Answers
1. 25
2. 6
3. 4

1. $(4^2-3^2)^2-\sqrt{2^5+4^3+5^2}-\sqrt{2^3+5^3+6^2}=$

   $(16-9)^2-\sqrt{32+64+25}-\sqrt{8+125+36}=$

   $7^2-\sqrt{121}-\sqrt{169}=49-11-13=25$

2. $16^3=4{,}096$ and $20^3=8{,}000$

   When we look at $5th$ powers of the natural numbers:

   $1^5=1,\ 2^5=32,\ 3^5=243,\ 4^5=1024,\ 5^5=3125,\ 6^5=7776$

   So 6 is the first possible candidate.

   Since $7^5=16{,}807>8{,}000$ then 6 is the answer.

3. $\left(\sqrt[3]{\frac{8}{27}}\right)^{-2}*\left(\frac{1}{2}\right)^{-4}*\left(\sqrt[4]{81}\right)^{-1}-2^{\sqrt{2^3+1^3}}=$

   $\left(\frac{2}{3}\right)^{-2}*2^4*3^{-1}-2^{\sqrt{8+1}}=$

   $\frac{3^2}{2^2}*\frac{2^4}{3}-2^3=3\cdot 2^2-8=4$

Category 5 – Algebra

1. How many integers *do not* satisfy the inequality below?

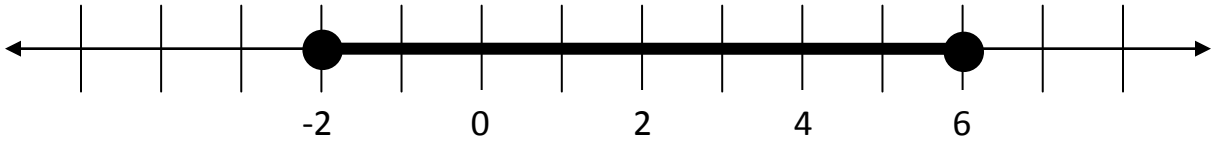
   $|3 - x| > 5$

2. Find the positive difference between the two solutions to the equation:

$$\left|\frac{2 \cdot x}{3} + 4\right| = 7$$

3. The graph below describes the solution to the inequality: $|x - A| \leq B$

   Find the value of $A + B$



Answers
1. _____
2. _____
3. _____

Solutions to Category 5 – Algebra

Answers
1. 11
2. 21
3. 6

1. Let's solve the inequality: $|3 - x| > 5$

   If the argument is positive we get:

   $3 - x > 5 \; or \; x < -2$

   If the argument is negative we get: $3 - x < -5 \; or \; x > 8$, so the solution to the inequality is $\{x < -2 \; or \; x > 8\}$. What integers do not fall in this range?

   $\{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$ - a total of 11 integers.

2. In the positive case: $\frac{2 \cdot x}{3} + 4 = 7$ we get $\frac{2 \cdot x}{3} = 3$ or $x = 4\frac{1}{2}$

   In the negative case: $\frac{2 \cdot x}{3} + 4 = -7$ we get $\frac{2 \cdot x}{3} = -11$ or $x = -16\frac{1}{2}$

   The differnce between the two solutions is 21.

3. The graph depicts all the points on the number line whose distance from 2 is no more than 4. In other words, it is the visualization of: $|x - 2| \leq 4$, which makes $A + B = 2 + 4 = 6$.

   The abosolute value function measures the distance between points on the line.

Category 6

1. The width of a rectangle is one inch more than its height. Its perimeter measures 82 inches. How many inches does its diagonal measure?

2. A common design for a soccer ball is made out of 20 Hexagons and 12 Pentagons. How many vertices does it have?

3. Which natural number N gives the greatest value to the expression: $N^{(11-N)}$?

4. Evaluate: $99_{base\,10} + 88_{base\,9} + 77_{base\,8} + 66_{base\,7} + 55_{base\,6} =$?
   Express your answer in base 10.

5. N is an integer that is a solution to: $|N - 5| = 7$, but is not a solution to: $|N - 7| = 5$. What is the value of N?

6. Using the values you obtained in questions 1 through 5, evaluate the folowing expression:
   $$\frac{A + C + D - E}{B}$$

Answers

1. _____inches=A
2. _____=B
3. _____=C
4. _____=D
5. _____=E
6. _____

Solutions to Category 6

Answers
1. 29
2. 60
3. 4
4. 325
5. -2
6. 6

1. If the height is H, then the width is $H + 1$ and the perimeter is $4 \cdot H + 2 = 82$ and so the height H is 20 inches and the width is 21 inches. The diagonal measures $\sqrt{20^2 + 21^2} = 29$ inches.

2. Though each Pentagon has five vertices, and each Hexagon has six, as can be gleaned from the picture, each vertex is shared by three polygons. In all we get: $\frac{5 \cdot 12 + 6 \cdot 20}{3} = 60$ vertices.
   Having flat faces, this solid is called a truncated icosahedron, but of course the ball's faces are puffed out and it's much closer to an ideal ball.

3. For $N = 11$, $N^{(11-N)} = 1$ and for greater value of N the expression is less than 1, so we only need look at smaller values for N:
   $1^{10} = 1\,, 2^9 = 512\,, 3^8 = 6561\,, 4^7 = 16384\,, 5^6 = 15625\,, 6^5 = 7776\,,$
   $7^4 = 2401\,, 8^3 = 512\,, 9^2 = 81\,, 10^1 = 10$
   We observe the greatest value for $N = 4$

4. $99_{base\ 10} + 88_{base\ 9} + 77_{base\ 8} + 66_{base\ 7} + 55_{base\ 6} =$
   $(10^2 - 1) + (9^2 - 1) + (8^2 - 1) + (7^2 - 1) + (6^2 - 1) =$
   $100 + 81 + 64 + 49 + 36 - 5 = 325$

5. First, let's solve $|N-5|=7$. The solutions here are $N=12$ and $N=-2$.
   Then the solutions for $|N-7|=5$ are $N=12$ and $N=2$.
   Since we're looking for a number that solves the first equation but not the second, that number is $N=-2$.

6. $\frac{A+C+D-E}{B}=\frac{29+4+325-(-2)}{60}=\frac{360}{60}=6$