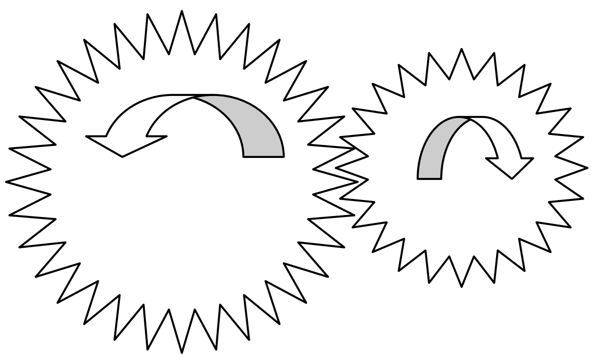
Intermediate Mathematics League of Eastern Massachusetts

Category 1 – Mystery

1. A recipe for making 16 pancakes requires - among other ingredients - the following: $1\frac{1}{2}$ cups of flour, 2 eggs, and 1 cup of milk.
   If you have 5 cups of flour, 5 eggs, and 5 cups of milk, then what is the largest number of pancakes you can make according to this recipe?

2. Two cogwheels are touching each other as in the illustration below.
   The smaller cogwheel has 25 teeth, and the bigger has 40 teeth.
   Note that the illustration does not show the correct number of teeth.
   How many complete turns will the smaller cogwheel perform until both wheels are back in the same position (same teeth touching each other)?



3. A square and a rectangle have the same area.
   The ratio between the lengths of the rectangle's sides is $4:1$.
   What percentage of the rectangle's perimeter is the square's perimeter?

Answers

1. ____________ pancakes
2. ____________ rotations
3. ____________ %

Solutions to Category 1 – Mystery

Answers

1. 40
2. 8
3. 80%

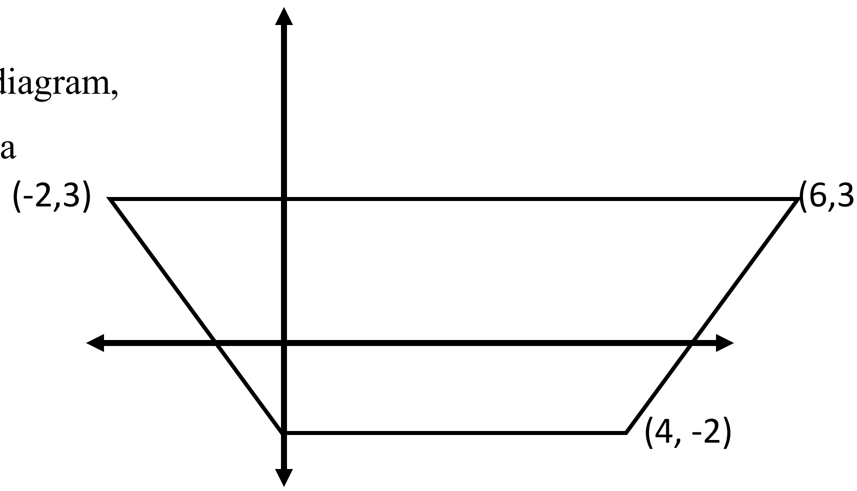
1. Following the recipe means we must use the ingredients in the same ratios as in the original recipe. Looking at these numbers, the limiting factor is the number of eggs (since for each egg we use less than 1 cup of each of the other ingredients, and we have them with identical quantities on hand). So if we followed the recipe using 5 eggs, we'd be multiplying all amounts by $2\frac{1}{2}$ and making $16 * 2\frac{1}{2} = 40$ pancakes.

2. With each revolution, the small wheel advances 25 teeth, and in order to realign with the bigger wheel the number of teeth advanced has to be a multiple of 40 (which is one revolution for the big wheel). The LCM of 25 and 40 is 200, which will be 8 revolutions for the smaller wheel (and 5 for the big wheel). (In the general case of wheels with M, N teeth, the wheels have to go $N/GCD = LCM/M$ and $M/GCD = LCM/N$ revolutions respectively).

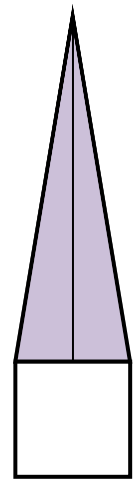
3. If we call the rectangle's sides R and $4 \cdot R$ (since we know the ratio), then its area is $4 \cdot R^2$ and therefore the square's side is $2 \cdot R$ (as it has the same area). The rectangle's perimeter is $2 \cdot (R + 4 \cdot R) = 10 \cdot R$, while the square's perimeter is $4 \cdot 2 \cdot R = 8 \cdot R$, which is 80% of the rectangle's.

Category 2 – Geometry

1. Given the coordinates in the diagram, how many units are in the area of the trapezoid?



2. The shaded triangle and square in the diagram share one side. The square's perimeter is 24 square inches, and its area is $\frac{2}{3}$ the area of the triangle. How many inches are in the triangle's height?



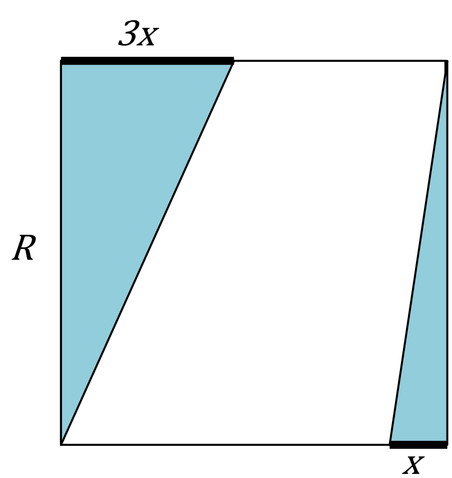
3. The diagram shows a square whose side is R, inside of which are two right triangles whose short legs are x and $3 \cdot x$.
   The white trapezoid's area is 70% of the square's area.
   Express $\frac{x}{R}$ as a percent.

Answers

1. __________ $units^2$

2. __________ inches

3. __________ %



Solutions to Category 2 – Geometery

Answers
1. 30
2. 18
3. 15%

1. The area of a trapezoid is given by:

   $\frac{1}{2} * Height * (Sum\ of\ bases)$

   From the diagram we can observe the height (the distance between the horizontal bases) is 5 units, the small base measures 4 units, and the large base measures 8 units. So the area is $\frac{1}{2} * 5 * (4 + 8) = 30$ units.

2. The square's perimeter is 24 inches, so its side measures 6 inches. Its area then is $6^2 = 36\ in^2$, and so the triangle's area is $36 * \frac{3}{2} = 54\ in^2$.

   If its height is H then $\frac{1}{2} * 6 * H = 54$ and so $H = 18$ inches.

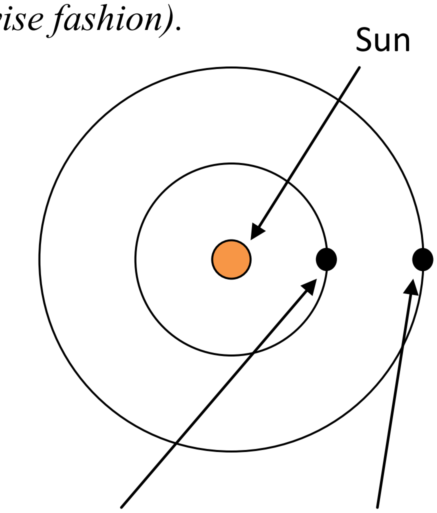
3. The triangles' combined area is $\frac{1}{2} \cdot R \cdot (x + 3 \cdot x) = 2 \cdot R \cdot x$ and so the remaining trapezoid's area is $R^2 - 2 \cdot R \cdot x = 70\% \cdot R^2$

   So we know that $2 \cdot R \cdot x = 30\% \cdot R^2$, or $2 \cdot x = 30\% \cdot R$, or $x = 15\% \cdot R$.

   (Here $R \neq 0$ of course, so we could divide by R).

Category 3 – Number Theory

1. It takes the planet Mercury 88 days to orbit the Sun, while the planet Venus orbits it in 224 days *(Both orbit in a counter-clockwise fashion).*
   If the two planets are aligned as in the diagram, how many orbits will Venus complete until they're aligned similarly again? (where both planets are directly to the right of the Sun).



2. N is a natural number for which:
   a. The GCF of N and 56 is 4
   b. the LCM of N and 56 is 280

   What is the value of N?
   ($GCF \equiv Greatest\ Common\ Factor,\qquad LCM \equiv Least\ Common\ Multiple$)

3. $1, 2, 3, 4, 6, 7, 12$ are the smallest seven factors of a natural number N.
   What is the least number of factors N can have?

Answers
1. __________ orbits
2. _________________
3. __________ factors

Solutions to Category 3 – Number Theory

Answers

1. 11
2. 20
3. 12

1. Note that during each orbit of Venus, there's a moment when the 3 objects are on one line, but the planets are not necessarily to the right of the Sun as in the diagram. What we're looking for is the time when both planets complete a whole number of orbits simultaneously, which happens every LCM of their orbital periods.
   $LCM(88\,,224) = LCM(2^3 \cdot 11\,, 2^5 \cdot 7) = 2^5 \cdot 7 \cdot 11$
   For Venus, the LCM is 11 times greater than its orbital period.
   (This would be 28 *orbits for Mercury).*

2. It is first necessary to factor $56 = 8 \cdot 7 = 2^3 \cdot 7$
   From $GCF(N, 56) = 4$ we conclude that 2^2 is a factor of N (no higher power, and no 7, but maybe other primes). From $LCM(N, 56) = 280 = 5 \cdot 56$ we conclude that 5 is a factor of N. No other factors are possible, or the LCM would have been greater. So $N = 2^2 \cdot 5 = 20$.
   Alternatively, we can directly find $N = \frac{GCF(N,56)*LCM(N,56)}{56} = \frac{4*280}{56} = 20.$

3. From the list of smallest factors, we can see that $2^2, 3, 7$ are all factors of N. We know that the prime factors 2 and 3 do not appear with higher powers (or 8 and 9 would have been in the list of smallest factors). So at a minimum, we have $N = 2^2 \cdot 3 \cdot 7 = 84$, though N may include higher powers of 7 and/or larger primes. The least number of factors is in the case where no other primes are included, of course, and in this case N has $(2 + 1) \cdot (1 + 1) \cdot (1 + 1) = 12$ factors.

Category 4 – Arithmetic

1. Express $\frac{1}{5} + \frac{1}{3}$ as a decimal.

   Use bar notation where appropriate.

2. Express $3.\overline{39}$ as a common fraction.

   A common fraction is of the form $\frac{\boldsymbol{m}}{\boldsymbol{n}}$ *where* $\boldsymbol{m}$ *and* $\boldsymbol{n}$ *share no common factors.*

3. In a certain town, 50% of the population are children, and 48% of children are girls.
   Half the boys, and a third of the girls, like to watch SpongeBob on TV.
   Assuming no adults do, what percentage of the population likes SpongeBob?

Answers

1. _______________

2. _______________

3. ____________ %

Solutions to Category 4 - Arithmetic

Answers
1. $0.5\overline{3}$
2. $\frac{112}{33}$
3. 21%

1. $\frac{1}{5}+\frac{1}{3}=\frac{3+5}{15}=\frac{8}{15}=0.5\overline{3}$

   This can be verified with long division.

2. If we call $x \equiv 3.\overline{39}$ then we can have $100 \cdot x = 339.\overline{39}$, and subtracting one from the other we get $99 \cdot x = 336$ or $x = \frac{336}{99} = \frac{112}{33}$

3. The overall percentage of boys is $50\% * 52\% = 26\%$, and the overall percentage of girls is $50\% * 48\% = 24\%$.

   Half the boys would be 13% of the population, and a third of the girls would be 8%, for a total of 21% SpongeBob fans.

Category 5 – Algebra

1. If you add 30 to the number A, you'd get a number that's 4 times greater than one third of A. What is A?

2. Inheriting a large sum of money, Mr. Lazy decided he does not need to work and can simply live off his fortune. He spends the same amount of money every year. After 14 years, he realized he has 75% of the money he had 2 years earlier. How many years will his fortune last overall?

3. A car drives from point A to point B, then turns around and drives back to point A at twice the original speed.
The average speed for the round trip was 64 mph (miles-per-hour).
What was the car's original speed?

Answers

1. ________________

2. ___________ years

3. ___________ mph

Solutions to Category 5 - Algebra

Answers

1. 90
2. 20
3. 48

1. Writing this algebraically:

   $A + 30 = 4 \cdot \frac{A}{3}$. To solve we can multiply both sides by 3:

   $3 \cdot A + 90 = 4A$ or $A = 90$

2. If we note the number of years the money will suffice for as M, then each year he spends $\frac{1}{M}$ of his initial fortune, and we know that:

   $1 - \frac{14}{M} = 75\% \cdot (1 - \frac{12}{M})$. Multiplying by $4 \cdot M$ we get:

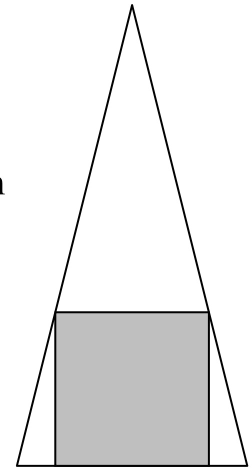
   $4 \cdot M - 56 = 3 \cdot M - 36$ or $M = 20$

3. If we note the car's original speed as V, and the distance between points A and B as D, then the trip from A to B took time $\frac{D}{V}$ and the trip back took time $\frac{D}{2 \cdot V}$.

   Overall the round-trip took time $\frac{D}{V} + \frac{D}{2 \cdot V} = \frac{3 \cdot D}{2 \cdot V}$ to cover a distance of $2 \cdot D$, and so the average speed is $\frac{2 \cdot D}{\frac{3 \cdot D}{2 \cdot V}} = \frac{4}{3} \cdot V = 64\ mph$ and so $V = \frac{3}{4} \cdot 64 = 48\ mph$.

   Note that the distance D does not affect the answer: For any distance, if we drive it once at 48 *mph, and then back at* 96 *mph, the average speed is* 64 *mph. 64 is the harmonic mean of 48 and 96:* $\frac{1}{64} = \frac{1}{2} \cdot (\frac{1}{48} + \frac{1}{96})$

Category 6 – Team round



1. The drawing shows a square inscribed inside an isosceles triangle.
   The triangle's height is 3 times the height of the square, and its width is 1.5 times the square's.
   If the square's perimeter is 8 inches, then how many square inches are in the triangle's area?

2. How many factors (including 1 and itself) does the number 960 have?

3. 2, 3, and 5 are the only prime factors of a natural number N.
   Overall, N has 30 factors. What is the least possible value of N?

4. A floor is made of square and rectangular tiles. The rectangular tile's length is 80% that of the square, and it takes 5 rectangular tiles to cover the same area as 3 square tiles. What percentage of the square's side is the rectangle's width?

5. It takes Bob twice as long as it takes Alice to pick a basket of apples.
   When they work together, they can fill the basket in 30 minutes.
   How many minutes will it take Alice working alone?

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

   $$B - \frac{A}{E} * \frac{D * C}{4}$$

   Note D is percentage!

Answers

1. ________in^2 = A
2. ______factors = B
3. ____________ = C
4. _________% = D
5. _______min = E
6. ____________ = F

Answers
1. $9 \quad = A$
2. $28 \quad = B$
3. $720 = C$
4. $75\% = D$
5. $45 \quad = E$
6. 1

Solutions to team round

1. Since the square's perimeter is 8 inches, its side measures 2 inches. That makes the triangle's width 3 inches and its height 6 inches, with an area of $\frac{3\cdot 6}{2} = 9\ in^2$.

2. The factorization of 960 is $960 = 2^6 \cdot 3 \cdot 5$, and therefore it has $(6+1)\cdot(1+1)\cdot(1+1) = 28$ facotrs

3. Since we know all of N's prime factor, we know that N is of the form: $2^a \cdot 3^b \cdot 5^c$, and since N has 30 factors we also know that: $(a+1)\cdot(b+1)\cdot(c+1) = 30 = 2\cdot 3\cdot 5$ (factorization of 30). *Note that we can safely assume that none of the powers a, b, c is zero, as we know that $2,3,5$ are indeed prime factors of N.* So we conclude that $\{a,b,c\} = \{1,2,4\}$ in some unknown order. The least possible value for N will match the case where we assign the greatest power to the smallest prime: $N = 2^4 \cdot 3^2 \cdot 5 = 720$.

4. If the square's side is x, then the rectangle's length is $0.8 \cdot x$ and we know that: $3 \cdot x^2 = 5 \cdot 0.8 \cdot x \cdot Width$. If the width is some portion of the square's side, we can write this as: $3 \cdot x^2 = 5 \cdot 0.8 \cdot p \cdot x^2$ where p is that portion. solving we get $p = \frac{3}{5\cdot 0.8} = \frac{3}{4} = 75\%$

5. If it takes Alice T minutes to fill the basket alone, then it takes Bob $2 \cdot T$ minutes. So each minute Bob fills $\frac{1}{2 \cdot T}$ of a basket, and Alice fills $\frac{1}{T}$ of a basket. We know that working together it takes them 30 minutes to fill one basket, so we can write this as: $30 \cdot \left(\frac{1}{T} + \frac{1}{2 \cdot T}\right) = 1$ so solve $T = 45$ minutes.

6. $B - \frac{A}{E} * \frac{D * C}{4} = 28 - \frac{9}{45} * \frac{75\% * 720}{4} = 28 - \frac{1}{5} * \frac{540}{4} =$
$28 - 27 = 1$