Intermediate Mathematics League of Eastern Massachusetts

Statistics and notes – not part of the original meet

Scheduled Meet Date	Oct. 20, 2011	
Number of Teams Competing	68	
Average Team Score	86	
Average Individual Score	7.1	(out of 18)

Category	1 Myst	2 Geom	3 NumTh	4 Arith	5 Alg
Number of Regulars Competing in This Category	406	403	401	402	405

Percent of Regulars with each possible score in the category:

0	19%	**29%**	**35%**	**38%**	**40%**
2	**42%**	24%	31%	34%	28%
4	25%	26%	20%	26%	13%
6	14%	21%	14%	2%	19%

A "Common Fraction" has only a numerator and a denominator, both of which are whole numbers. The numerator may be greater than the denominator. It must be in lowest terms.

Category 1 – Mystery

1. Over summer vacation, Jenna worked as a lifeguard 40 hours a week and earned \$10 an hour. Her friend Nicky worked at the grocery store making only \$9 an hour, but worked 45 hours a week. After 8 weeks, how much money do they have together?
   !!! Whole dollar amounts can be written with or without decimal places. Other amounts must be written with two decimal places. !!!

2. In a class of 20 students, the teacher wanted to take pictures of pairs of students. How many pictures does the teacher have to take in order to have pictures of all possible student pairs?

3. The letters F, G, H represent mathematical operations on natural numbers according to these definitions ($Where\ N\ is\ any\ natural\ number$):
   $F(N) = 4 \times N + 3$
   $G(N) = N \div 2 + 5$
   $H(N) = N \times (N + 1)$
   Find the value of the expression $F(G(H(4^2 - 4)))$

Answers

1. \$_____
2. _____Pictures
3. _____

Solutions to Category 1 – Mystery

Answers
1. $6,440.00 or $6,440
2. 190
3. 335

1. Jenna earned a total of:
   $\$10_{per\ hour} \times 40_{hours} \times 8_{weeks} = \$3{,}200$
   Nicky earned a total of:
   $\$9_{per\ hour} \times 45_{hours} \times 8_{weeks} = \$3{,}240$
   So together they have a total of $6,440.

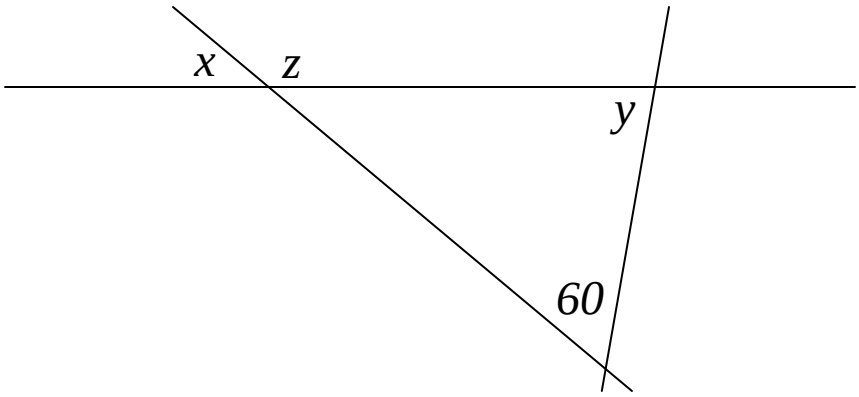
2. Each student has to appear in 19 pictures (one with each one of the other 19 students in class), so that would suggest $20 \cdot 19 = 380$ pictures, but we counted each possible pair twice (once for each of the two students in the picture), so we have to divide this number by 2. The answer then is 190 pictures.

3. Using the given definitions:
   $F\left(G\left(H(4^2-4)\right)\right) = F\left(G\left(H(12)\right)\right) = F\left(G(12 \times 13)\right) = F\left(G(156)\right) =$
   $= F(156 \div 2 + 5) = F(83) = 4 \times 83 + 3 = 335$

Category 2 – Geometry

1. The supplementary angle to a given $\measuredangle A$ measures 4 times as much as $\measuredangle A$'s complementary angle. How many degrees are there in the measure of $\measuredangle A$?

2. In the drawing below, the measure of one angle is given as 60 degrees, and the measure of the angle '*x*' is half as much as the measure of the angle '*y*'. How many degrees are there in the measure of the angle '*z*'?



3. The angle between the hour-hand and the minute-hand on a clock at 3 o'clock is 90 degrees. What is it 20 minutes earlier? *(Measure from the hour hand to the minute-hand counter-clockwise).*

Answers
1. _____ degrees
2. _____ degrees
3. _____ degrees

Solutions to Category 2 – Geometry

Answers
1. 60
2. 140
3. 200

1. If we write this information algebraically, then we know that: $180 - A = 4 \cdot (90 - A)$. This we can simplify to: $180 - A = 360 - 4 \cdot A$ and rearrange to: $3 \cdot A = 180$ to find $A = 60$. Alternatively, we can engage in a little trial and error to find out the value of A.

2. The missing angle in the triangle equals x (opposite angles), and so we know that $x + y + 60 = 180$ as the three angles in a triangle. Given that x is half as much as y, we can rewrite this as $3 \cdot x + 60 = 180$ to find that $x = 40$ degrees. z is the supplement to x and so equals 140 degrees.

3. It takes the minute hand an hour to complete a revolution, so 20 minutes earlier, it was 120 degrees back (pointing at the numeral 8). The hour hand progresses at a pace of only 30 degrees every hour, so in 20 minutes it only progresses 10 degrees. The answer then is $90 + 120 - 10 = 200$ degrees.

Category 3 – Number Theory

1. What is the smallest natural number which is divisible by $3,4,5,6,and\ 7$?

2. What is the smallest 5-digit number that is a multiple of 18?

3. What is the sum of all factors of the number 1,000? *(including 1 and itself).*

Answers
1. _____
2. _____
3. _____

Solutions to Category 3 – Number Theory

Answers

1. 420
2. 10008
3. 2340

1. In order to be divisible by 3 and 4, a number has to be a multiple of $3 \cdot 4 = 12$. Adding the requirements for 5 and 7, we have to multiply by $5 \cdot 7 = 35$, so we get $12 \cdot 35 = 420$. We don't have to worry about the 6, as any multiple of 12 is sure to be a multiple of 6.

2. The answer is one of the numbers in the range $\{10{,}000 - 10{,}017\}$.
   In order to be divisible by 18, our number has be divisible by both 2 and 9, so we're looking for an even number with a sum of digits that divides by 9, and the smallest such number possible is $10{,}008 = 18 \times 556$.

3. $1{,}000 = 1 \times 1{,}000 = 2 \times 500 = 4 \times 250 = 5 \times 200 = 8 \times 125 = 10 \times 100 = 20 \times 50 = 25 \times 40$
   The sum of factors then is:
   $1 + 2 + 4 + 5 + 8 + 10 + 20 + 25 + 40 + 50 + 100 + 125 + 200 + 250 + 500 + 1000 = 2{,}340$

Category 4 – Arithmetic

1. Find the value of the expression: $\frac{1}{2} + \frac{3}{4} \cdot \frac{8}{9}$

   Give your answer as a decimal rounded to the nearest hundredth.

2. The following table lists the scores distribution on a recent test:

Score	*Number of students who got that score*
50	2
60	7
70	6
80	9
90	4
100	2

   What is the positive difference between the *mean* score and the *median* score?

3. Jeff wrote down a list of natural numbers, all different from each other.

   He noticed the following properties of the list:

   - The mean value was 20.
   - The median was one of the numbers in the list.
   - The sum of all numbers in the list was 4 times the median.

   What is the largest possible number of numbers in the list if the median is a two-digit number?

Answers

1. _____

2. _____

3. _____

Solutions to Category 4 – Arithmetic

Answers
1. 1.17
2. 1
3. 5

1. $\frac{1}{2}+\frac{3}{4}\cdot\frac{8}{9}=\frac{1}{2}+\frac{3}{4}\cdot\frac{4\cdot2}{3\cdot3}=\frac{1}{2}+\frac{2}{3}=\frac{3+4}{6}=1\frac{1}{6}=$

   $=1.16666\ldots\cong1.17$

2. According to the table, there are a total of 30 students/scores. 15 students scored 70 or below, and 15 scored 80 or above, so the ***median*** is 75. To calculate the ***mean*** score:

$$Mean=\frac{2\cdot50+7\cdot60+6\cdot70+9\cdot80+4\cdot90+2\cdot100}{30}$$

$$=\frac{10\cdot(10+42+42+72+36+20)}{30}=\frac{222}{3}=74$$

   The difference between the *median* and the *mean* then is $75-74=1$.

3. Let N be the number of numbers in the list, M be the value of the median, and S be the sum of the numbers in the list. Since the mean is 20, S must be 20N. If N is even, then M could not be in the list because M would be halfway between the two middle numbers, and not be either one. So, N is odd. Also, from the problem, S=4M. So S=20N=4M or 5N=M. If N=5, then M=25, so one list could be 1,9,25,30,35, so we know 5 is a possible number of numbers in the list. If N=7, then S=140 and M=35. Suppose the list in increasing order starts with 1,2,3,35, which are the smallest possible. The last 3 numbers in the list are greater than 35, but they must add to 99, which is impossible. If the first 3 numbers were larger, then the sum left over is even smaller than 99, so it is always impossible to have N=7. It gets worse for larger lists, so the largest possible N is **5**.

   Note: The original solution had 19 as the answer, but only 5 was accepted as correct.

Category 5 – Algebra

Express your answers as common fractions (not decimals) if necessary.

1. Find the value of x that makes the following equality true:

   $\frac{x+2}{2} + \frac{x-2}{3} = \frac{x}{2}$

2. Find the value of M that will make the equation below an identity:

   $M \cdot x + 2 \cdot M \cdot (x+1) + 3 \cdot M \cdot (x+2) = 3 \cdot x + 4$

3. We define the operation ♣ as follows: $A♣B \equiv A^2 - B$

   Find the value of a natural number N for which: $8♣(N♣4) = 7♣17$

Answers
1. _____
2. _____
3. _____

Solutions to Category 5 - Algebra

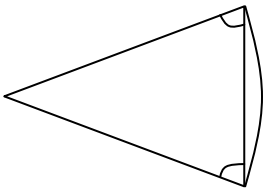
Answers
1. -1
2. $\frac{1}{2}$
3. 6

1. Multiplying the equation by the common denominator, 6, we get:
   $3 \cdot (x+2) + 2 \cdot (x-2) = 3 \cdot x$, or -
   $3 \cdot x + 6 + 2 \cdot x - 4 = 3x$, which we can rearrange into –
   $2 \cdot x = 4 - 6 = -2$, or $x = -1$

2. If we simplify the left side of the equation:
   $M \cdot x + 2 \cdot M \cdot (x+1) + 3 \cdot M \cdot (x+2) =$
   $M \cdot x + 2 \cdot M \cdot x + 2 \cdot M + 3 \cdot M \cdot x + 6 \cdot M =$
   $6 \cdot M \cdot x + 8 \cdot M = 2 \cdot M \cdot (3 \cdot x + 4)$ and this has to equal the right side of the equation, which is $3 \cdot x + 4$, so $2 \cdot M = 1$, or $M = \frac{1}{2}$

3. $8\clubsuit(N\clubsuit4) = 8^2 - (N\clubsuit4) = 64 - (N^2 - 4) = 68 - N^2 = 7\clubsuit17 = 7^2 - 17 = 32$
   So we're left with: $68 - N^2 = 32$, or $N^2 = 36$, and since N is a natural number, the solution is $N = 6$

Category 6



1. At Nancy's birthday party, Nancy's mother cut a big round pizza equally for all the children (One slice each).
   She noticed that if she connected each pizza slice's corners to form a triangle, then the base angles measure 78 degrees each. How many children were at the party?

2. If we list all the primes that are less than 100, from the smallest to the greatest, then there are only two whose value is the average (mean) of their neighboring primes. The mean of these two is also a prime. What is it?

3. There are two natural numbers (N) for which the *mean* and the *median* of the numbers $\{N,\ 3 \cdot N,\ N^2\}$ equal each other.
   What is the sum of these two natural numbers?

4. A loaded truck drove from point A to point B at 50 miles-per-hour (*mph*). After unloading its load, it drove back on the same route at 60 *mph*, and the trip back was two minutes shorter. How many miles are there between point A and point B?

5. A, B, C, D, E, F represent digits (all different from each other).
   Given the following equalities:
   $ABCDEF \times 2 = CDEFAB$
   $ABCDEF \times 3 = BCDEFA$
   $ABCDEF \times 4 = EFABCD$
   $ABCDEF \times 5 = FABCDE$
   $ABCDEF \times 6 = DEFABC$
   What is the value of $(A + B + C + D + E + F)$?

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:
   $$D - \frac{A - C}{B - E}$$

Answers

1. _____=*A*
2. _____=*B*
3. _____=*C*
4. _____*miles*=*D*
5. _____=*E*
6. _____

Answers

1. 15
2. 29
3. 7
4. 10
5. 27
6. 6

Solutions to Category 6

1. The angle at the point of each slice has to complete the triangle to 180 degrees, and so will measure $180 - 2 \cdot 78 = 24$ degrees.
   Since all the slices complete a circle, all the point angles must add up to 360 degrees, so there must be $360 \div 24 = 15$ slices, and kids.

2. The list of primes is: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$. If we look at all triplets of neighboring numbers, we see that only 5 and 53 are the average of their neighboring primes, and their mean is 29, also a prime.

3. The *mean* value of the list is always $\frac{N+3\cdot N+N^2}{3}$.
   The *median* depends on the respective values: for all natural numbers, N would be the smallest number of the three.
   If we assume that $3 \cdot N < N^2$(as you might from the order in which the list was given), then the *median* would be $3 \cdot N$, and equating it to the *mean* we get $N^2 = 5 \cdot N$, the natural solution for which is $N = 5$. If we assume that $3 \cdot N > N^2$, then the *median* is N^2, and equating it to the mean we get $N = 2$. Lastly, In the case where $3 \cdot N = N^2$ the list is $\{3, 9, 9\}$ and the *median* (9) and *mean* (7) are not the same. The requested sum is $5 + 2 = 7$.

4. Remember that $Distance = Speed \times Time$ so if we note the time the initial trip took as T, we can write: $50 \cdot T = 60 \cdot (T - \frac{2}{60})$ [Recall we measure time in hours, so 2 minutes are $\frac{1}{30}$ of an hour]. The solution is $T = \frac{1}{5}$ hour, and the distance is $50 \cdot T = 10$ miles.

5. Some of you will recognize the cyclical pattern of $\frac{1}{7} = 0.\overline{142857}$ immediately.

   If not, there's some trial involved:

   Any 6-digit number greater than 166666 times 6 will result in a 7-digit number, so we conclude that $A = 1$ and $B \leq 6$.

   From the order of products, we can conclude that $A < C < B < E < F < D$ (looking at the first digit of each product), and clearly B, D, and C are even.

   Since the 3rd multiple starts with B, then B cannot be 6 (since our original number is less than 166666), so $B \leq 4$, and since it has to be even and greater than C, it has to be 4.

   Looking at the 4th multiple, we know that CD is a multiple of 4, and since $A < C < B < E < F < D$ then C is at least 4 less than D, which leaves us only '28' or '48' as possibilities. So $D = 8$, and since $C < B = 4$, C has to be 2.

   So far we know $A = 1, B = 4, C = 2, D = 8$ and so E, F have to be picked from $\{5,6,7\}$.

   Since the 5th multiple ends with 'E', then $E = 5$.

   Lastly, the sum of digit has to divide by 3, and so F has to be 7.

   The final sum is $1 + 4 + 2 + 8 + 5 + 7 = 27$

6. $D - \frac{A-C}{B-E} = 10 - \frac{15-7}{29-27} = 10 - \frac{8}{2} = 10 - 4 = 6$