Intermediate Mathematics League of Eastern Massachusetts

Category 1 – Mystery

1. Two thirds of all students at school are girls.
   Two thirds of all students have dark hair.
   If half the boys have dark hair, then what portion of the girls has black hair?
   Express your answer as a common fraction.

2. There are 9 calories in each gram of fat, and 4 calories in each gram of protein or carbohydrates (carbs). A cup of low-fat milk contains 2.5 grams of fat, 13 grams of carbs, and 8 grams of proteins.
   How many cups of milk do you have to drink in order to get 2,000 calories?
   Express you answer as a decimal, rounded to the nearest tenth.

3. The *Body-Mass-Index* (BMI) is a measure associated in medicine with the risk for heart disease. Its definition is: $\boldsymbol{BMI = \frac{Weight}{Height^2}}$
   where the weight is measured in *Kilograms* and the height in *Meters*.
   Given that $1\ Kilogram = 2.2\ Lbs$ and $1\ Inch = 2.54\ \ Centimeters$, what is the ***BMI*** of a person whose height is 6' (six feet) and whose weight is 176 lbs?
   Round your answer to the nearest whole number.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1 – Mystery

Answers
1. $\frac{3}{4}$
2. 18.8
3. 24

1. Since boys are one-third of students, the half of them with dark hair amounts to one-sixth of all students.
   We know that overall two-thirds of students have dark hair, so that means that the number of dark-haired girls have to be $\frac{2}{3} - \frac{1}{6} = \frac{1}{2}$ of *all* students.
   Since the girls represent two-thirds of all students, it would take three-quarters of them to amount to half of all students ($\frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$).

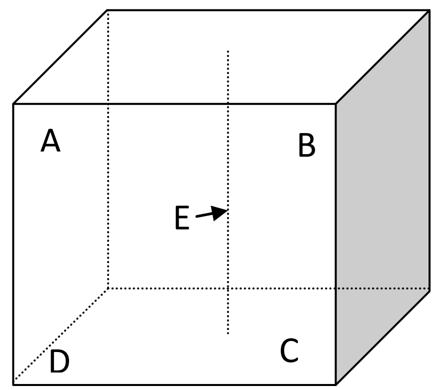
2. Each cup of milk will have:
   $2.5 * 9 = 22.5$ calories from fat, and $(13 + 8) * 4 = 84$ calories from carbs and proteins, for a total of 106.5 calories per cup.
   In order to get 2,000 calories we have to drink $\frac{2{,}000}{106.5} = 18.779 \ldots \cong 18.8$ cups.

3. Translating the weight to metric units:
   $176\ lbs = \frac{176}{2.2}\ kg = 80\ kg$
   Translating the height:
   $6\ feet = 6 \cdot 12\ inches = 72\ inches = 72 \cdot 2.54\ cm = 182.88\ cm \cong 1.83m$
   $$BMI = \frac{Weight}{Height^2} = \frac{80}{1.83^2} = 23.88 \cong 24$$
   A *BMI* higher than 25 is associated with increased cardiac risk.

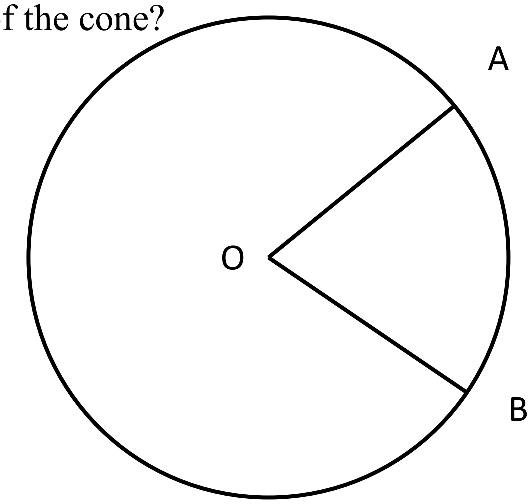
Category 2 – Geometry

1. What is the surface area of a cylinder with a radius of 3 feet and a height of 10 feet? *Include the top and bottom of the cylinder, use* $\pi = 3.14$, *and round your answer to the nearest whole square foot.*

2. If you take a cube whose sides all measure 1 inch, and connect all four corners (A, B, C, D) on one face of the cube to the cube's center, E, then how many cubic inches are there in the volume of the resulting pyramid $ABCDE$?
Express your answer as a common fraction.



3. The radius OA of the circle below measures 10 inches.
We cut off a sector AOB with a central angle $\measuredangle AOB = 72\ degrees$.
We then roll the remaining (larger) shape into a cone.
How many inches are there in the height of the cone?



Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 2 – Geometery

Answers
1. 245
2. $^1/_6$
3. 6

1. The area is the outside plus the top/bottom:

   $2 \cdot \pi \cdot R \cdot H + 2 \cdot \pi \cdot R^2 = 2 \cdot \pi \cdot R \cdot (H + R) =$

   $2 \cdot 3.14 \cdot 3 \cdot (10 + 3) = 244.92 \cong 245_{sq\ feet}$

2. Two ways to see this: The cube's center E is half the distance between the face $ABCD$ and the opposing face, so its distance from face $ABCD$ (the pyramid's height) is ½ inch. And the volume is the area of $ABCD$ multiplied by the height, divided by 3, so $\frac{1}{6}$. Thought of differently, the cube has 6 faces, so we can build 6 such pyramids, covering all of the cube's volume, so a pyramid's volume is one-sixth of the cube's volume (which is one cubic inch).

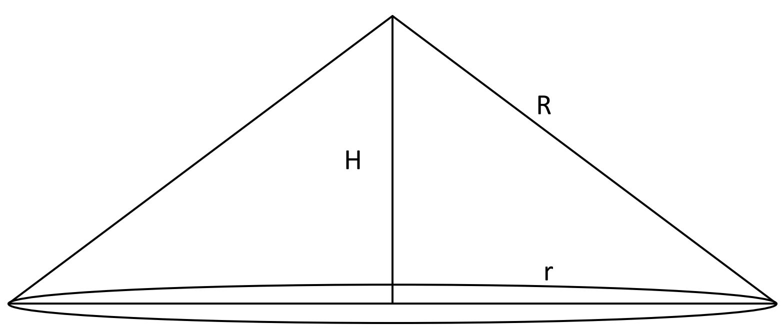
3. Since we cut off a fifth ($\frac{72}{360}$) or the circle, then the cone's base will obey:

   $2 \cdot \pi \cdot r = \frac{4}{5} \cdot 2 \cdot \pi \cdot R$ and so $r = \frac{4}{5} \cdot R = 8\ inches.$

   The cone's height can be calculated using Pythagoras:

   $H^2 = R^2 - r^2 = 10^2 - 8^2 = 36$, and so $H = 6\ inches$.

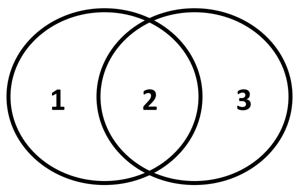
   The larger the angle cut, the taller the cone.



Category 3 – Number Theory

1. Set A contains 10 elements. The union $A \cup B$ contains 20 elements, while the intersection $A \cap B$ contains 4 elements.
   How many elements are the in the set B?

2. In a general Venn diagram of 2 sets, we can have 3 different areas, as shown in this diagram:



   Where the common area in the middle represents the intersection of the two sets. How many different areas are possible in a Venn diagram of 4 sets?

3. How many subsets of the set $\{A, B, C, D, E, F\}$ contain at least one vowel?

Answers
1. ________________
2. ________________
3. ________________

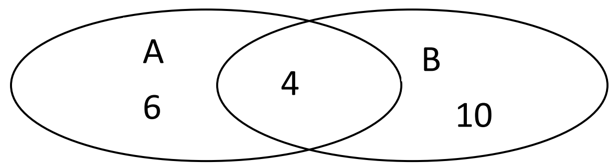
Solutions to Category 3 – Number Theory

Answers	
1.	14
2.	15
3.	48

1. The number of elements in a union of two sets is the sum of the number of elements in each one, minus the number of elements in their intersection. Denoting the number of elements in a set A as $|A|$, we can write this as:

   $$\left|A \bigcup B\right| = |A| + |B| - \left|A \bigcap B\right|$$

   Using the numbers in our case we get $|B| = 20 + 4 - 10 = 14$. We can see the number of elements in this Venn diagram:



2. There are 4 areas containing only one set each.
   There are 6 areas containing the intersections of exactly two sets each (AB, AC, AD, BC, BD, CD), 4 areas of intersection of exactly 3 sets (ABC, ABD, ACD, BCD), and one area of the intersection of $ABCD$.
   Overall 15 distinct areas.
   This is basically the number of subsets of $\{A, B, C, D\}$ without the empty set.

3. The set $\{A, B, C, D, E, F\}$ has 6 elements and therefore $2^6 = 64$ subsets.
   Removing the 2 vowels $\{A, E\}$ we're left with 4 elements that make $2^4 = 16$ subsets with no vowel in them. So the remaining 48 subsets must contain at least one vowel.
   Note we were careful to count the empty subset with the no-vowel group.

Category 4 – Arithmetic

1. If Larry tosses a fair coin four times, what is the probability he'll get the same result each time? *Express your answer as a common fraction.*

2. How many possible passwords can there be that match the listed criteria?
   a. The password must be either 2 or 3 characters long.
   b. The password may contain only letters and numbers.
   c. The password is case-sensitive (So 'Math' is not identical to 'MATH').

   (The same character can be used more than once in a password).

3. Three players bring two balls each to a tennis practice. When the practice is over, each player picks two balls at random. What is the probability that each player ended up with the same two balls she came with?
   Express your answer as a common fraction.

Answers

1. ________________

2. ________________

3. ________________

Solutions to Category 4 – Arithmetic

Answers

1. $\frac{1}{8}$

2. 242172

3. $\frac{1}{90}$

1. Each individual toss has two possible outcomes (Head / Tail), and so four consecutive tosses have 2^4 possible outcomes.
   Two of these fit our condition (four consecutive tails, and four consecutive heads), so the probability is $\frac{2}{2^4} = \frac{1}{8}$

2. The characters allowed are the 26 UPPER-CASE letters, the 26 lower-case letters, and the 10 digits, so a total of 62 characters.
   There are therefore 62^2 passwords made up of two characters, and 62^3 made up of three, for a total of $62^2 + 62^3 = 242{,}172$ passwords.

3. This seems quite difficult. If we label the players A, B, C and number the balls $1,2,3,4,5,6$, then the *random picking at the end is equivalent to* arranging the balls in some order, and giving the first two to player A, the next two to player B, etc. And so the question becomes: If we arrange the six balls in a random order, what is the probability that balls #1, #2 are in the first two positions, balls #3, #4, are in the third and fourth positions etc.
   There are of course $6!$ permutations for the balls, and $2^3 = 8$ of them match our criteria [Arrangements of the sort $\{(1,2), (3,4), (5,6)\}$ where we don't care about the internal order in each pair, as long as the pairs are in the correct order. Each pair has two permutations, hence 2^3].
   The probability then is $\frac{2^3}{6!} = \frac{8}{720} = \frac{1}{90}$

More solutions to Category 4 – Arithmetic

We wanted to show two slightly different ways of solving these problems, to demonstrate that there can be more than one way to arrive at a correct solution.

1. Regardless of the result of the first toss (Head or Tail), the second toss has a $\frac{1}{2}$ probability of being the same, the third toss again has a $\frac{1}{2}$ probability of being the same, and so does the fourth. Overall we get $(\frac{1}{2})^3 = \frac{1}{8}$ as before.

3. Here's another way to think of this: The first player has a $\frac{1}{3}$ probability that the first ball she picks is hers (because 2 out of the 6 balls are), and then a $\frac{1}{5}$ probability that the second ball she picks is hers.
   *Assuming the first player did in fact pick her own balls**, the second player has a $\frac{1}{2}$ probability that the first ball she picks is hers, and then a $\frac{1}{3}$ probability that the second ball is hers.
   Multiplying all these probabilities (independent events) we get:
   $\frac{1}{3} \cdot \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{90}$
   ** Otherwise the second player may face different probabilities.*

Category 5 – Algebra

1. If we add 4 inches to a square's length, and shorten its width by 4 inches, we get a new rectangle whose area is 75% of the original square's area.
   How many inches are in the length of the square's side?

2. One of the solutions to the equation $x^2 + B \cdot x - 8 = 0$ is $x = 2$.
   What is the value of the other solution?

3. The product of the solutions of the equation $x^2 - 9x + C = 0$, is twice as much as their sum. What is the larger of the two solutions?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5 - Algebra

Answers	
1.	8
2.	-4
3.	6

1. If we call the square's length x, then we can write:

   $(x+4)\cdot(x-4) = x^2 - 16 = \frac{3}{4}\cdot x^2$

   and so $x^2 = 64$ and $x = 8\ inches$. *Since we're looking for a length, we're interested only in the positive solution.*

2. Since we know that $x = 2$ is a solution, we can use this value in the original equation to get $2^2 + B\cdot 2 - 8 = 0$ and solve to get $B = 2$.

   Now the original equation is known to be $x^2 + 2\cdot x - 8 = 0$ and the other solution for this is $x = -4$.

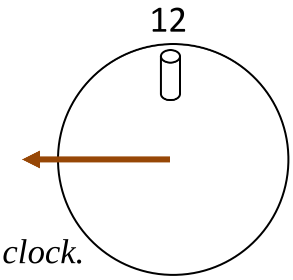
3. In the general case of a quadratic equation $A\cdot x^2 + B\cdot x + C = 0$ we know that the sum of solutions is $-B$, and their product is $A\cdot C$. In our case $A = 1$ and $B = -9$, and so the sum of solutions is $+9$, and their product is $18 = C$.

   Our equation then is $x^2 - 9\cdot x + 18 = (x-3)\cdot(x-6) = 0$.

   The solutions are of course $x = 3$ and $x = 6$.

Category 6

1. How many different numbers can be written with the digits {1,2,3,4,5}?
   Each digit may only be used once in each number. (So $1, 12, 31$ *are ok,* 16 *or* 11 *are not).*



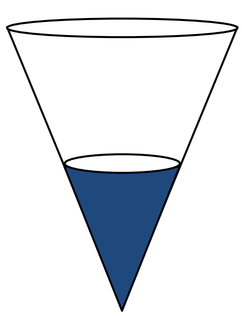
2. A car's wheel's radius is 16 inches. The valve stem currently points at '12' (using a clock-like system). Where will it point to after the car moves 31.4 feet to the left? *Use* $\pi = 3.14$ *and give your answer as an hour on the clock.*

3. Rain falling at a constant rate fills an (initially empty) upside-down cone up to a height of 1 foot in 1 hour. How many *more* hours until the cone is filled to a height of 2 feet? *See diagram at bottom of page.*

4. Three tennis balls are stacked in a cylindrical package. If we assume that the radius of the cylinder is the same as the radius of the balls, and its height equals three diameters, then what fraction of its volume is 'wasted'? i.e. not taken by the balls? *Express your answer as a a common fraction.*

5. When tossing a fair coin 12 times, what is the probability you get 'Tails' no more than 4 times? *Express your answer as a percent, rounded to the nearest whole number percentage.*

6. Using the values you obtains in questions 1 through 5, evaluate the following expression:
   $$\frac{D \cdot (A - 100 \cdot E) + B}{3 \cdot C}$$
   Note that E is a percent.



Answers

1. ______________=*A*
2. ______________=*B*
3. ______________=*C*
4. ______________=*D*
5. ____________%=*E*
6. ________________

Solutions to Category 6

1. We can make numbers with 1,2,3,4, or 5 digits.
   How many single-digit numbers? Obviously 5.
   How many 2-digit numbers? 5 choices for first digit, and 4 choices for the second, or $\frac{5!}{3!} = 20$ 2-digit numbers. etc.
   The total is:
   $$\frac{5!}{4!} + \frac{5!}{3!} + \frac{5!}{2!} + \frac{5!}{1!} + 5! = 5 + 20 + 60 + 120 + 120 = 325$$
   Note there's the same number (120) of 5-digit numbers, as there is of 4-digit numbers (because there are 5 different choices for the 4 digits, and only one choice for the five).

2. The car moves to the left 31.4 feet, which is
   $(31.4 \cdot 12") = (120 \cdot \pi)\ inches$.
   The wheel's circumference is $2 \cdot \pi \cdot R = (32 \cdot \pi)\ inches$.
   So over the trip the wheel spins 3 whole rotations, plus an amount equivalent to $24 \cdot \pi\ inches$ or an additional $\frac{24}{32} = \frac{3}{4}$ whole rotation. So at the end, the valve stem has shifted ¾ rotation counter-clockwise, and so will point at 3 o'clock.

3. Let's use lower-case letters for the smaller cone, and upper-case letters for the large one. Their volumes then are $\frac{\pi \cdot r^2 \cdot h}{3}$, and $\frac{\pi \cdot R^2 \cdot H}{3}$. We know that $H = 2 \cdot h$ and therefore (because the cross-section of the cone will yield similar triangles) $R = 2 \cdot r$ and so $\frac{\pi \cdot R^2 \cdot H}{3} = \frac{\pi \cdot (2 \cdot r)^2 \cdot 2h}{3} = 8 \cdot \frac{\pi \cdot r^2 \cdot h}{3}$ i.e. the larger cone's volume is eight times the smaller one's, and so it'll take an *additional* seven

Answers	
1.	325
2.	3
3.	7
4.	$\frac{1}{3}$
5.	19%
6.	5

hours to fill at the same rate.

4. Since the balls are stacked on each other, then the answer does not depend on how many balls there are, as long as the cylinder is the same height as the stack. If we have a ball of radius R then its volume is $\frac{4}{3} \cdot \pi \cdot R^3$. A cylinder with a radius R and height of $2 \cdot R$ has a volume of $(\pi \cdot R^2) \cdot (2 \cdot R) = 2 \cdot \pi \cdot R^3$, and so the ball takes only $\frac{4/3}{2} = \frac{2}{3}$ of the cylinder's volume. One-third is 'wasted'.

5. Tossing a coin 12 times we have $2^{12} = 4{,}096$ different outcomes, and we need to know in how many the number of 'Tails' is four or less.

   There are ${}_0C_{12}$ outcomes with 0 tails, ${}_1C_{12}$ outcomes with 1 tail, etc.

   The total number of outcomes we look for then is:

   ${}_0C_{12}+{}_1C_{12}+{}_2C_{12}+{}_3C_{12}+{}_4C_{12}= 1 + 12 + 66 + 220 + 495 = 794$

   The probability then is $\frac{794}{4096} = 0.1938 \ldots \cong 19\%$

   By symmetry, this is also the probability the number of tails is 8 or more, and so we're left with about 62% *probability that the number of tails is* $5, 6,$ *or* 7.

6. $\frac{D \cdot (A - 100 \cdot E) + B}{3 \cdot C} = \frac{\frac{1}{3} \cdot (325 - 19) + 3}{3 \cdot 7} = \frac{102 + 3}{21} = 5$