

Intermediate  
Mathematics League  
of  
Eastern Massachusetts

Meet #4      February 2011

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Category 1 – Mystery

1. Barrel *A* contains 10 gallons of water and 2 gallons of oil, barrel *B* contains 3 gallons of water and 4 gallons of oil, and barrel *C* contains 1 gallon of oil. When the contents of all are mixed together, what is the percentage of oil in the mixture?

2. Three plastic circles touch each other in such a way that their diameters all lie on the same line.

The distance  $\overline{AB}$  measures 48 inches. How many inches are in the circumference of the resulting shape?

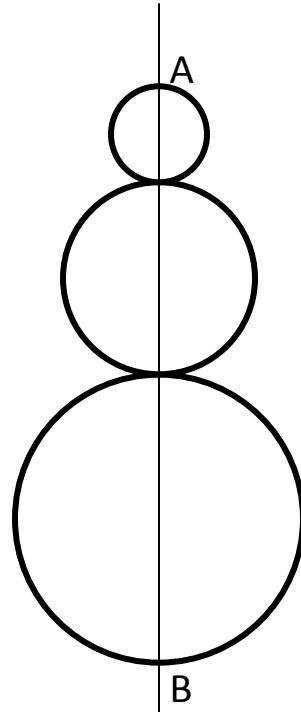
*Use  $\pi = 3.14$  and round your answer to the nearest integer.*

3. Given the following estimate for the U.S. population:

- Every 8 seconds a new baby is born
- Every 15 seconds a person dies
- Every 36 seconds an immigrant comes to the U.S.
- Every 200 seconds an emigrant leaves the U.S.

What is the net change in U.S. population during the 2 hour period of this meet?

*(For example, if you think it grew by 2 write +2, and if you think it decreased by 2 write -2).*



Answers

1. \_\_\_\_\_ %
2. \_\_\_\_\_
3. \_\_\_\_\_

Solutions to Category 1 – Mystery

1. When we add all barrels together we end up with 13 gallons of water and 7 gallons of oil, and so the oil's share of the mixture is  $\frac{7}{7+13} = \frac{7}{20} = 35\%$
  
2. Though the three circles have different radii, in each one of them the circumference is  $\pi$  times the diameter, of course, so the total circumference is  $\pi$  times the sum of diameters, which is simply the height, or  $\pi \cdot 48 = 150.72 \dots \cong 151 \text{ inches.}$
  
3. We can calculate the impact of each the these effects separately, and then add them all up:

In 2 hours (7,200 seconds) we get:

- $\frac{7200}{8} = 900 \text{ new babies}$
- $\frac{7200}{15} = 480 \text{ deaths}$
- $\frac{7200}{36} = 200 \text{ new immigrants}$
- $\frac{7200}{200} = 36 \text{ emmigrants leaving}$

Adding up we get  $900 - 480 + 200 - 36 = +584$ .

We can say that the net effect is an increase of one person every

$$\frac{7200}{584} \cong 12.33 \text{ seconds.}$$

*[These numbers are not completely accurate!]*

Answers

1. 35%
2. 151
3. +584

Category 2 – Geometry

1. The Earth's radius is 3,960 miles, and the Moon's radius is 1,080 miles.

What is the difference (in miles) in the length of their equators?

*Assume both are perfect spheres, use  $\pi = 3.14$  and round to the nearest whole number.*

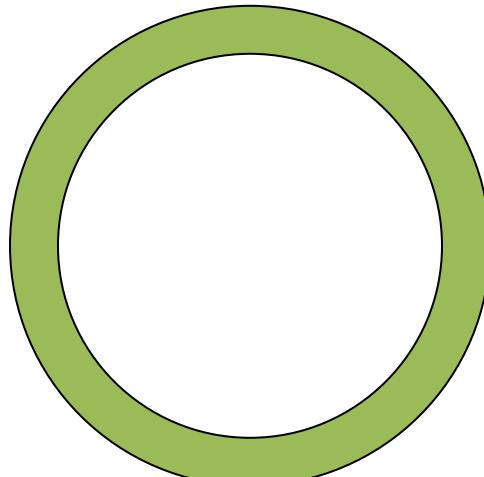
2. A square is inscribed inside a circle whose radius is 1 inch.

How many square inches are in the area of the square?

*Inscribed means that the square's corners all touch the circle from the inside.*

3. The shaded ring-shaped area measures 36% of the outer circle's area.

If the radius of the outer circle is 10 inches, then how many inches are there in the radius of the inner circle?



Answers

1. \_\_\_\_\_

2. \_\_\_\_\_

3. \_\_\_\_\_

Solutions to Category 2 – Geometry

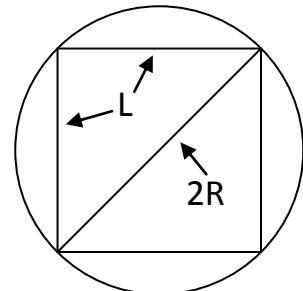
Answers

1. Since a circle's circumference (the equator) is given by the formula  $C = 2 \cdot \pi \cdot R$  then the difference is:

$$2 \cdot \pi \cdot (3,960 - 1,080) = 6.28 * 2,880 = 18,086.4 \\ \cong 18,086 \text{ miles}$$

1. 18,086
2. 2
3. 8

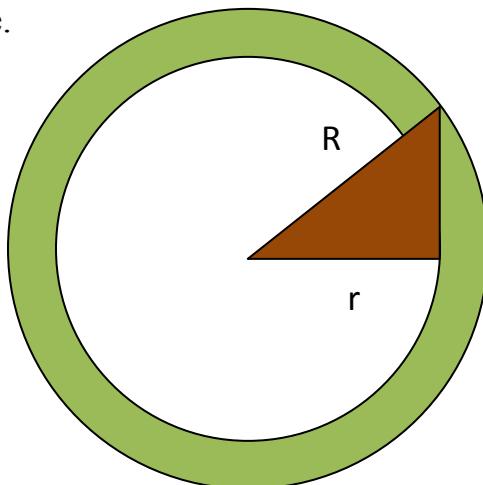
2. Calling the circle's radius  $R$  and the square's side  $L$ , we use Pythagoras to see that  $L^2 + L^2 = (2 \cdot R)^2 = 4$ , and the square's area is simply  $L^2 = 2 \text{ square inch}$



3. Let's call the radii of the large and small circles  $R$  and  $r$  respectively. The ring's area is the difference in the circles' area, so we know that:  
 $\pi \cdot (R^2 - r^2) = 36\% \cdot \pi \cdot R^2$  or in different words,  $r^2 = 64\% \cdot R^2 = 64 \text{ in sq}$   
and so  $r = 8 \text{ inches}$ .

It's worth seeing the relationship to the Pythagorean theorem here:

The area of the ring is the same as the area of a circle whose radius is the other leg of the triangle.



Category 3 – Number Theory

1. What is the sum of the first 100 numbers in the series:

$$\{-93, -85, -77, -69, \dots\}$$

2. A team of robots starts building a tower at 1pm.

It takes them one hour to construct the first floor, and each consecutive floor takes twice as long as the preceding one.

What time is it when they're done building the 6<sup>th</sup> floor?

*Be sure to note AM or PM in your answer!*

3. When snow falls we measure its height on a clean surface, but when new snow falls then it weighs down the snow underneath it, so the height of a snow pile is actually less than the sum of accumulation.

Let's assume that *each* new layer of 1" (1 inch) of snow compresses *all* the snow below it by a factor of 10%. (*So for example, if 2" of snow fall, then the bottom 1" will be compressed to 0.9", and the top 1" will still be 1"*).

If overall 12" (twelve storms of 1" each) of snow fall , then how many inches are in the height of the snow pile?

*Round your answer to the nearest hundredth of an inch.*

Answers

1. \_\_\_\_\_

2. \_\_\_\_\_

3. \_\_\_\_\_

Solutions to Category 3 – Number Theory

Answers

1. 30,300

2. 4am

3. 7.18

1. The  $N^{th}$  term in this series is given by  $-93 + 8 \cdot (N - 1)$ , and so the sum of the first 100 terms is:

$$S_{100} = (a_1 + a_{100}) * 50 = 50 * (-93 - 93 + 8 * 99) = \\ = 50 * (606) = 30,300$$

2. Since each floor takes twice as long as the preceding one, the total time required is:  $1 + 2 + 4 + 8 + 16 + 32 = 63 \text{ hours} \equiv 15(\text{mod } 24)$   
15 hours past 1pm will be 4am.

3. The top layer is still 1" thick, the second layer is  $0.9 \cdot 1"$  thick, and so on, until the bottom layer which is  $0.9^{11} \cdot 1"$  thick.

The sum of these heights is a sum of a geometric series:

$$\text{Sum} = 1" \cdot (1 + 0.9 + 0.9^2 + \dots + 0.9^{11}) = 1" \cdot \frac{1 - 0.9^{12}}{1 - 0.9} = 1" \cdot \frac{0.71757 \dots}{0.1} \cong 7.18"$$

Category 4 – Arithmetic

1. The tab for your dinner party at a restaurant came to \$160. You want to leave a 20% tip, and also need to add 6.25% tax on the price of dinner. How much does the dinner cost altogether? (There is no tax on the tip, and assume you do not pay a tip on the tax either.)
  
2. If the price of a gallon of milk increases by 25%, and the price of a gallon of oil decreases by 15%, then both prices will be the same.  
What percent of the price of (a gallon of) oil is the price of (a gallon of) milk?
  
3. Your company, **IMLEM Lemonade Inc.**, has purchased \$700 worth of supplies (lemons, sugar, paper cups). This is enough to make 3,000 cups of lemonade.
  - You sold 80% of that number, for \$0.50 per cup.
  - From the money you have left (the amount you received from customers minus the cost of your supplies), you pay 40% in salaries.
  - From the amount left, you pay 20% sidewalk-tax to your parents.

How much money do you have left?

| Answers     |
|-------------|
| 1. \$ _____ |
| 2. _____ %  |
| 3. \$ _____ |

### Solutions to Category 4 - Arithmetic

1. The tax amount is  $\$160 * 6.25\% = \$10$

The tip amount is  $\$160 * 20\% = \$32$

Overall:  $\$160 + \$10 + \$32 = \$202$

*Editor's note: The original problem did not say there was no tip on the tax. This would give \$204, which was not allowed as an answer.*

### Answers

1. \$202 (\$202.00)

2. 68%

3. \$240

2. If we name the respective prices  $A$  and  $B$ , then we know that:

$A + 25\% \cdot A = B - 15\% \cdot B$  or:  $125\% \cdot A = 85\% \cdot B$  and so:

$$A = \frac{85}{125} \cdot B = 68\% \cdot B$$

3. Our revenue (the amount we got from customers) is:

$\$0.50_{per\ cup} * 80\% * 3,000_{cups} = \$1,200$

Minus the cost of supplies, you have \$500 left.

After paying salaries, you have left  $60\% \cdot \$500 = \$300$

After paying tax, you have left  $80\% \cdot \$300 = \$240$

Category 5 – Algebra

1. If I add 9 to my secret number, I'll get a number that is 90% more than the number I'll get if I subtract 9 from my secret number.

What is my secret number?

2. Donald and Mickey both collect stamps.

Donald figured he needs 30% more stamps in order to have half as many as Mickey does.

Mickey figured that 12 more stamps would make his collection 3 times the size of Donald's. How many stamps do they have altogether?

3. You want to fill an empty tank with 50 gallons of water and 50 gallons of oil.

At your disposal are two barrels with different mixtures that you may use to fill the empty barrel:

- Barrel A contains a mixture of 80% water and 20% oil.
- Barrel B contains a mixture of 40% water and 60% oil.

How many gallons of mixture from barrel A will you use?

Answers

1. \_\_\_\_\_

2. \_\_\_\_\_

3. \_\_\_\_\_

Solutions to Category 5 - Algebra

1. Calling my secret number  $N$ , we know that:

$N + 9 = 190\% \cdot (N - 9)$ , which we can simplify into:

$$9 \cdot (1 + 1.9) = N \cdot (1.9 - 1) \text{ or } 0.9 \cdot N = 9 \cdot 2.9$$

$$\text{So } N = \frac{9 \cdot 2.9}{0.9} = 29$$

Answers

1. 29  
2. 108  
3. 25

2. If Donald has  $D$  stamps, and Mickey has  $M$  stamps, we know that:

$$D + 30\% \cdot D = 1.3 \cdot D = \frac{1}{2} \cdot M$$

$$M + 12 = 3 \cdot D$$

So we can write the second equation as  $2.6 \cdot D + 12 = 3 \cdot D$ , so  $0.4 \cdot D = 12$ , or  $D = 12 \cdot 2.5 = 30$  and  $M = 3 \cdot D - 12 = 78$ . So together  $D + M = 108$ .

3. If we are to use  $G$  gallons from barrel  $A$ , then it will contain  $80\% \cdot G$  gallons of water. We will use  $(100 - G)$  total gallons from barrel  $B$ , to bring the total volume to 100 gallons, and these will contain  $40\% \cdot (100 - G)$  gallons of water, and so we require that  $80\% \cdot G + 40\% \cdot (100 - G) = 50$  gallons, and the solution for this is  $G = 25$  gallons.

*Check:* the 25 gallons from  $A$  will be 20 gallons water and 5 gallons oil, and the 75 gallons from  $B$  will be 30 gallons water and 45 gallons oil.

It is insightful to summarize the given information in this problem like this:

|         | Barrel A | Barrel B |
|---------|----------|----------|
| Water % | 80       | 40       |
| Oil %   | 20       | 60       |

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### Category 6

1. The sum of the first six terms of an arithmetic series is four times as much as the sum of the first three terms. The first term is 7. What is the third term?

2. What is the smallest natural number N that satisfies both conditions:

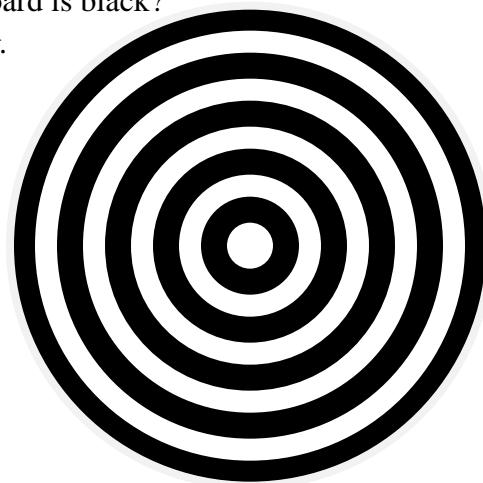
$$3 \cdot N \equiv 1 \pmod{5}$$

$$5 \cdot N \equiv 1 \pmod{7}$$

3. Old MacDonald wanted to fence a circular vegetable patch to keep pests away. The fence would cost him \$5 per yard, and then it would cost him \$1 per square yard to fertilize, plow, seed, and water the patch. At the end of the season, he can sell his crop for \$2 per square yard. What's the smallest-integer radius he needs to fence to make at least \$1,000 profit? Use  $\pi = 3.14$

4. The picture below shows a dart-board made of concentric circles. The inner circle's radius is 1 inch, and each consecutive radius increases by 1 inch. What portion of the board is black?

*Express your answer as a percent.*



5. Evaluate the expression:  $66^{13} \pmod{7}$

6. Using the values you obtained in questions 1 through 5, evaluate the expression:

$$\frac{D \cdot 10 \cdot (B - 3 \cdot E)}{A - C}$$

### Answers

1. \_\_\_\_\_ = A

2. \_\_\_\_\_ = B

3. \_\_\_\_\_ = C

4. \_\_\_\_\_ % = D

5. \_\_\_\_\_ = E

6. \_\_\_\_\_

Solutions to Category 6

1. If the difference between any two consecutive elements is  $d$ , then the sum of the first three elements is  $(7 + 7 + 2 \cdot d) \cdot \frac{3}{2}$ , and the sum of the first six is  $(7 + 7 + 5 \cdot d) \cdot \frac{6}{2}$ , and so we're told that:  $(14 + 5 \cdot d) \cdot 6 = 4 \cdot 3 \cdot (14 + 2 \cdot d)$ . The solution for this is  $d = 14$ , and so the third element is  $7 + 2 \cdot 14 = 35$

2. The conditions are equivalent to saying:

$$3 \cdot N = 5 \cdot j + 1 \text{ AND } 5 \cdot N = 7 \cdot k + 1 \text{ for some integers } j, k.$$

From the first condition,  $j$  values of 1,4,7,10, ... will give integer solutions for  $N$ , and from these values, the first to give an integer solution  $k$  for the second condition is  $j = 10$ ,  $N = 17$ ,  $k = 12$ .

3. The cost of the fence is  $\$5 \cdot 2\pi \cdot R$ , and the net revenue from the patch would be  $\pi \cdot R^2 \cdot (\$2 - \$1)$ , and so to make at least \$1,000 the requirement is:

$$\$ \cdot \pi \cdot R^2 - \$10 \cdot \pi \cdot R > \$1,000 \text{ or } R^2 - 10 \cdot R > \frac{1,000}{\pi} \cong 318.47$$

And we're looking for the smallest integer  $R$  to satisfy this.

With trial and error you should get  $R = 24$  yards.

4. The area of the black rings can be calculated as a difference in consecutive circles' areas. Over all black portion is:

$$\begin{aligned} & \frac{(10^2 - 9^2) + (8^2 - 7^2) + (6^2 - 5^2) + (4^2 - 3^2) + (2^2 - 1^2)}{10^2} \\ &= \frac{19 + 15 + 11 + 7 + 3}{100} = \frac{55}{100} = 55\% \end{aligned}$$

Answers

1.  $35 = A$   
 2.  $17 = B$   
 3.  $24 = C$   
 4.  $55\% = D$   
 5.  $3 = E$   
 6.  $4$

5. Since  $66 \equiv 3 \pmod{7}$  then:

$$\begin{aligned}66^{13} &= 3^{13} \pmod{7} = (3^4)^3 \cdot 3 = 81^3 \cdot 3 \equiv 4^3 \cdot 3 = 64 \cdot 3 \\&= 192 \equiv 3 \pmod{7}\end{aligned}$$

We used the fact that  $81 \pmod{7} \equiv 4 \pmod{7}$

We could have also seen that  $64 \pmod{7} \equiv 1 \pmod{7}$

$$6. \frac{D \cdot 10 \cdot (B - 3 \cdot E)}{A - C} = \frac{55\% \cdot 10 \cdot (17 - 3 \cdot 3)}{35 - 24} = \frac{55\% \cdot 80}{11} = \frac{44}{11} = 4$$