Intermediate Mathematics League of Eastern Massachusetts

Category 1 – Mystery

1. At a recent field trip, a class of 30 students stopped for lunch and each student ordered either a hot dog ($1.75) or a pizza ($2). Overall, the students spent $56. How many students ordered pizza?

2. An Origami expert took a square piece of paper measuring 8×8 inches, and folded it in half, then in half again, and again…making five folds altogether. How many square inches are there in the area of the final shape?

3. What is the smallest number of coins you need to represent any sum up to (and including) $1?
No $1 or $½ coins are allowed.

Answers

1. ________________

2. ________________

3. ________________

Solutions to Category 1 – Mystery

Answers	
1.	14
2.	2
3.	10

1. If we call the number of students who ordered pizza N, then we know that: $\$2 \cdot N + \$1.75 \cdot (30 - N) = \$56$
   This simplifies to: $\$0.25 \cdot N = \$3.5$, or $N = 14$.

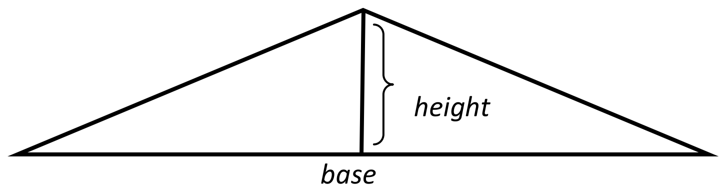
2. The area of the original piece is $8 \times 8 = 64$ square inches. Though the expert can fold it in different ways (creating rectangles, squares, or triangles), each fold cuts the area by half, so after five folds we've cut the area by $2^5 = 32$, so we end up with an area of $\frac{64}{32} = 2$ square inches.

3. Intuitively, we want to use big coins as much as possible.
   Therefore, we'll use 3 quarters, and need to find a way to represent 1-25 cents with the fewest coins possible:
   4 cents, 1 nickel, and 2 dimes will do the trick, for a total of 10 coins.
   (Note that the sum of our coins is more than $1).

Editor note: One student argued that the answer is 9, since you can indeed represent any sum up to $1.00 with 9 coins, just not the same 9 coins each time. This answer was debated but rejected because the answer with that interpretation may as well be 1 coin, since that is the <u>smallest number of coins</u> that can represent the value $0.01 (it does say "any sum" not "all sums.").
A question with the answer 9 – and only the answer 9 – would be difficult to write clearly.
The question as intended (one set of coins for all amounts) has a practical value (what coins should you put in your pocket to be able to pay any amount) and only one meaningful answer.

Category 2 – Geometry

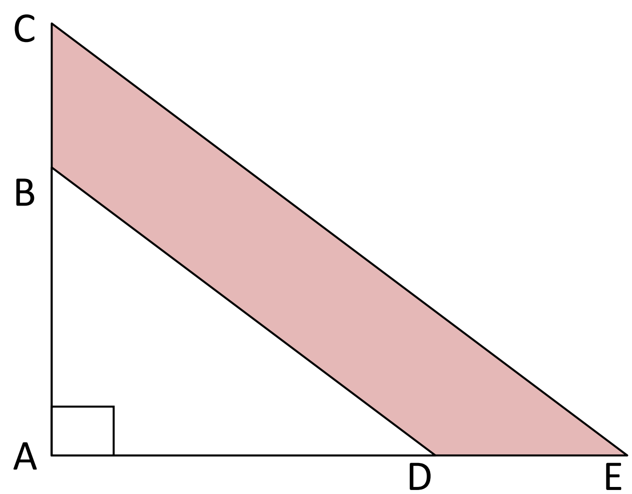
1. In the isosceles triangle below, the base measures 24 inches, and the height measures 5 inches. How many inches are in the triangle's perimeter?



2. Spongebob and Patrick each drew a regular polygon. Patrick's polygon had twice as many sides and six times as many diagonals as Spongebob's. How many sides to Patrick's polygon?

3. In the drawing below:
   - $\overline{BD}$ is parallel to $\overline{CE}$
   - $\overline{CB} = 3\ cm$, $\overline{DE} = 4\ cm$, and $\overline{BD} = 10\ cm$

How many centimeters squared are in the shaded trapezoid $BCED$?



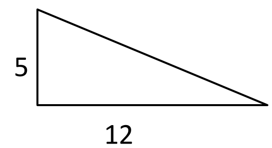
Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 2 – Geometery

Answers

1. 50
2. 12
3. 30

1. In each half of the triangle, we have $5^2 + 12^2 = 169 = Hypotenuse^2$ and therefore the Hypotenuse measures 13 inches, and the whole perimeter measures $13 + 13 + 24 = 50\ inches$.



2. If Spongebob's polygon has N sides, then it has $\frac{1}{2} \cdot N \cdot (N-3)$ diagonals. Patrick's polygon then has $2 \cdot N$ sides and $\frac{1}{2} \cdot (2 \cdot N) \cdot (2 \cdot N - 3)$ diagonals. So from the problem we know that: $(2 \cdot N) \cdot (2 \cdot N - 3) = \mathbf{6} \cdot N \cdot (N-3)$ which we can simplify to $(2 \cdot N - 3) = 3 \cdot (N-3)$. The solution is $N = 6$ and so Patrick's polygon has $2 \cdot N = 12$ sides.

3. Since $\overline{BD}$ is parallel to $\overline{CE}$ then the triangles ABD and ACE are similar. Therefore $\frac{\overline{AB}}{\overline{AD}} = \frac{\overline{BC}}{\overline{DE}} = \frac{3}{4}$ and since in triangle ABD we have $\overline{AB}^2 + \overline{AD}^2 = 10^2$ we get that $\overline{AB} = 6cm$ and $\overline{AD} = 8cm$. Therefore the area of triangle ABD is $24\ cm^2$ and the area of triangle ACE is $54\ cm^2$, and the trapezoid is their difference, $30\ cm^2$.

Category 3 – Number Theory

1. One gram of oxygen contains 1.88×10^{22} molecules.

   There are, of course, 1,000 grams in one kilogram.

   The Earth's atmosphere is estimated to weigh 5×10^{18} kilograms, of which 20% is oxygen. How many oxygen molecules are in the atmosphere?

   Express you answer in scientific notation.

2. Express 3.6×10^2 in Binary (Base 2).

3. Find the value of N that solves the equation below:

   $$100_{Base\ 4} \times 100_{Base\ 5} = 620_{Base\ N}$$

Answers

1. _______________

2. _______________

3. _______________

Solutions to Category 3 – Number Theory

Answers

1. 1.88×10^{43}

2. 101,101,000 or 101101000 (commas optional)

3. 8

1. There are $20\% \times (5 \times 10^{18})$ kilograms of oxygen, each one containing $1{,}000 \times (1.88 \times 10^{22})$ molecules, so overall: $0.2 \times 5 \times 1.88 \times 10^{3+18+22} = 1.88 \times 10^{43}$ oxygen molecules. *(If you don't believe, count for yourself!).*

2. $3.6 \times 10^2 = 360 = 256 + 64 + 32 + 8 = 2^8 + 2^6 + 2^5 + 2^3 =$ $101{,}101{,}000_{Base\ 2}$

3. $100_{Base\ 4} \times 100_{Base\ 5} = 4^2 \times 5^2 = 400 = 620_{Base\ N} =$ $= 6 \cdot N^2 + 2 \cdot N = 2 \cdot N \cdot (3 \cdot N + 1)$

   Trying a few values we can quickly realize that $N = 8$ is the solution.

   It is helpful to make these two observations:

   a. Since $400_{Base\ 10} = 620_{Base\ N}$ then we know $N < 10$
   b. We know that N is a factor of 400, since 620 ends in a '0'.

Category 4 – Arithmetic

1. Evaluate the following expression:

$$(3^3 - 2^5)^2 + \sqrt[3]{2^6} + \sqrt{9 \cdot 4^2}$$

2. N is a natural number that makes the equation below true. Find the value of N.

$$N^2 \cdot \left(\frac{3}{5}\right)^2 = \left(\frac{5^{-1} \cdot \sqrt{144}}{\sqrt[3]{8}}\right)^2 \cdot \sqrt{81}$$

3. M is a natural number greater than 1 such that the difference between the 4th and 3rd powers of M is 4 times the difference between the 3rd and 2nd powers. What is the value of M?

Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 4 - Arithmetic

Answers
1. 41
2. 6
3. 4

1. $(3^3 - 2^5)^2 + \sqrt[3]{2^6} + \sqrt{9 \cdot 4^2} =$

$(27 - 32)^2 + \sqrt[3]{64} + \sqrt{3^2 \cdot 4^2} =$

$(-5)^2 + 4 + 3 \cdot 4 = 25 + 4 + 12 = 41$

2. $N^2 \cdot \left(\frac{3}{5}\right)^2 = \left(\frac{5^{-1} \cdot \sqrt{144}}{\sqrt[3]{8}}\right)^2 \cdot \sqrt{81}$

$N^2 \cdot \frac{9}{25} = \left(\frac{12}{5 \cdot 2}\right)^2 \cdot 9$

$N^2 \cdot \frac{9}{25} = \left(\frac{6}{5}\right)^2 \cdot 9 = 6^2 \cdot \frac{9}{25}$

$N^2 = 6^2 \; so \; the \; natural \; solution \; is \; N = 6$

3. We know that $M^4 - M^3 = 4 \cdot (M^3 - M^2)$. Taking the common factors out in each side of the equation we get:

$M^3 \cdot (M - 1) = 4 \cdot M^2 \cdot (M - 1)$ so we can divide both sides by $M^2 \cdot (M - 1)$ to get $M = 4$.

Editor note: The original question did not have "greater than 1" and so M=1 was a debated but accepted answer since 0 = 4 times 0. Also accepted was 1,4.

Category 5 – Algebra

1. How many integers satisfy the inequality below?

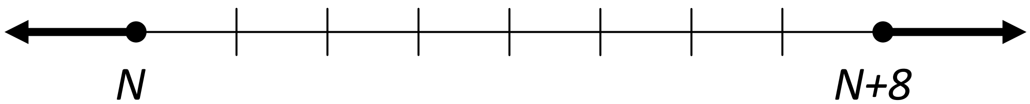
$$\left|\frac{N}{2}\right| - 12 < 4$$

2. The solution of the inequality $|x - M| \leq 2$ is given by: $3 \leq x \leq 7$.

   What is the value of the parameter M?

3. The solution to the inequality $|x + 1| \geq 4$ is given by the line graph below.

   What is the value of N?



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5 - Algebra

Answers
1. 63
2. 5
3. -5

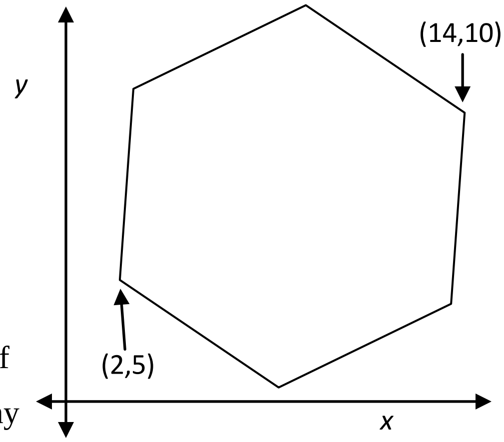
1. We can rewrite the inequality $\left|\frac{N}{2}\right| - 12 < 4$ as $\left|\frac{N}{2}\right| < 16$ or $|N| < 32$. This is true for the integers in the range $\{-31, -30, \dots., 0, \dots, 30, 31\}$ - a total of 63 integers.

2. When solving $|x - M| \leq 2$, if the argument inside the absolute value is positive we get $x \leq M + 2$, and if it is negative we get $x \geq M - 2$. Comparing this to the given solution $3 \leq x \leq 7$ we see that $M = 5$.

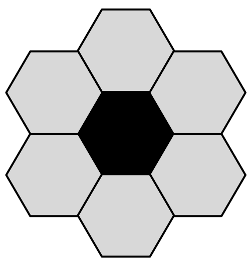
3. When solving $|x + 1| \geq 4$, if the argument is positive we get $x + 1 \geq 4$ *or* $x \geq 3$, and if it's negative we get $x + 1 \leq -4$ *or* $x \leq -5$. Comparing these to the line graph we conclude that $N = -5$.

Category 6

1. How many units are there in the perimeter of the regular hexagon in the diagram?
[The drawing is correct – the sides are not parallel to the y axis].



2. The drawing below is made of 7 regular hexagons.
If we continue to build two more concentric layers of hexagons, wrapped around this shape, then how many sides will there be in the overall perimeter?



3. N is a natural number such that 96 is one of the 49 factors of N^3.
What is the value of N?

4. What is the smallest number of sides possible in a regular polygon, if the number of diagonals is more than ten times the number of sides?

5. Solve the following base 4 problem, and express your answer in Decimal (base 10):

$$\frac{(10{,}230 - 2{,}230)}{10} = ?$$

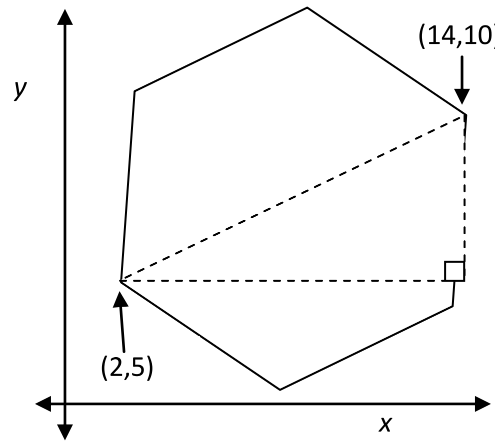
6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\frac{D}{(B-A)\cdot(C-E)}$$

Answers

1. ____________=A
2. ____________=B
3. ____________=C
4. ____________=D
5. ____________=E
6. ____________

Solutions to Category 6



1. If we connect the two vertices, then the length of the diagonal is $\sqrt{(14-2)^2+(10-5)^2}=$
$=\sqrt{144+25}=13$, and in a regular hexagon this equals twice the length of a side.
So the perimeter is 3 times that length, namely 39 units.

2. When drawing additional layers, you can notice that only those six hexagons that are directly above the central hexagon's sides have 3 of their sides exposed, and those in between them have only 2 sides exposed. The outer layer has 6 'corner' hexagons (with 3 exposed sides each), and 12 'side' hexagons (with only 2 sides exposed), for a total of $6\cdot 3+12\cdot 2=42$ sides.
(Each layer's perimeter is 12 sides longer than the preceding layer's).

3. $96=2^5\cdot 3$ is a factor of N^3. The factorization of N^3 should have its prime factors with powers that are multiples of 3, so at the minimum it'll be $2^6\cdot 3^3$, but this number only has $(6+1)\cdot(4+1)=28$ factors. To get 49 factors, N^3 has to equal $2^6\cdot 3^6$, and so $N=2^2\cdot 3^2=36$. (Recall that only perfect squares have an odd number of factors).

4. If there are N sides, then there are $\frac{N\cdot(N-3)}{2}$ diagonals, and so the requirement is $\frac{N\cdot(N-3)}{2}>10\cdot N\ or\ N^2-3\cdot N>20\cdot N$.
This translates to $N^2>23\cdot N, or\ N>23$.
So the minimal number is 24.

Answers

1. $A=39$
2. $B=42$
3. $C=36$
4. $D=24$
5. $E=32$
6. $F=2$

5. $10{,}230 - 2{,}230 = 10{,}000 - 2{,}000 = 2{,}000\ in\ base\ 4$.

(To see this, remember that $10{,}000_{base\ 4} = 4^4, and\ 2{,}000_{base\ 4} = 2 \cdot 4^3$).

As for the division $2{,}000 \div 10 = 200$ in any base, and translating to Decimal we get $200_{base\ 4} = 2 \cdot 4^2 = 32_{decimal}$.

6. $\frac{D}{(B-A)\cdot(C-E)} = \frac{24}{(42-39)\cdot(36-32)} = \frac{24}{3\cdot4} = 2$