You may use a calculator today!

Meet #5

March 2010

Intermediate

Mathematics League

of

Eastern Massachusetts

Meet #5

March 2010

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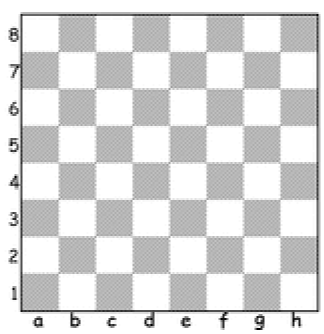
Category 1 - Mystery

Meet #5, March 2010

1. A model of a statue is built to a scale of 1: 5 from the same material as the real statue and weighs 4 pounds. How many pounds does the real statue weigh?

2. Two students are selected at random from a group of 3 girls and 3 boys. What is the probability that exactly one of each gender was selected? *Express your answer as a percent to the nearest whole number*.

3. Imagine a chessboard. Start by placing a coin in each one of the 64 squares.
   Then drop another coin in each square, except for the a1 square.
   Then a 3rd coin in each square except for those in the 2-by-2 square from a1 to b2.
   Then a 4th coin in each square except for those in the 3-by-3 square from a1 to c3.
   …
   Keep going until an 8th coin is placed in each square in row 8 and column h.
   How many coins are used altogether?



Answers	
1. _________	*pounds*
2. _________	%
3. _________	

Solutions to Category 1 - Mystery

Meet #5, March 2010

Answers

1. 500
2. 60%
3. 372

1. The weight is proportional to the volume of material.

   Since the model is built to a scale of $1:5$, its volume is $\frac{1}{5^3}$ that of the real airplane (Volume = Length × Width × Height for each piece).

   So the real statue's weight will be $4_{pounds} \times 5^3 = 4 \times 125 = 500_{pounds}$.

2. Intuitively, we can see that once the first student is selected, there now remain 2 of the same gender and 3 of the opposite gender, so the probability the second one is of the opposite gender is $\frac{3}{5} = 60\%$.

   Another way to arrive at the solution is to note that there ${}_2C_6 = \frac{6!}{2!\times 4!} = 15$ ways to select any two students, and exactly $3 \times 3 = 9$ ways to select one girl and one boy.

   The desired probability then is $\frac{9}{15} = \frac{3}{5} = 0.6 = 60\%$.

3. The number of coins in each individual square is shown in this table:

8	8	8	8	8	8	8	8
7	7	7	7	7	7	7	8
6	6	6	6	6	6	7	8
5	5	5	5	5	6	7	8
4	4	4	4	5	6	7	8
3	3	3	4	5	6	7	8
2	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8

   In order to sum we notice that we have 1 square of 1 coin, 3 squares of 2 coins, 5 squares of 3 coins, etc. The total is:

$$1 \cdot 1 + 3 \cdot 2 + 5 \cdot 3 + 7 \cdot 4 + 9 \cdot 5 + 11 \cdot 6 + 13 \cdot 7 + 15 \cdot 8 = 372$$

   There are many other interesting ways to sum up the coins – can you think of a few more?

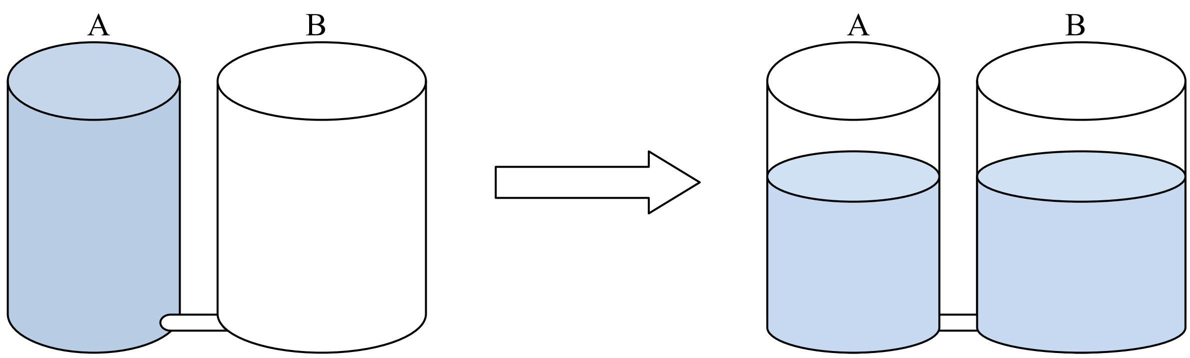
Category 2 - Geometry

Meet #5, March 2010

1. How many square feet are in the surface area of a cube with a volume of 8 cubic feet?

2. A ball with a radius of 3 inches has the same volume as a pyramid with a square base measuring 6×6 inches. How many inches are in the pyramid's height?
   Use $\pi = 3.14$ *and express your answer as a decimal to the nearest hundredth.*

3. Cylinder *A* has a radius of 3 inches, and is filled with water to a height of 5 inches. When connected by a small tube to cylinder *B* (which is initially empty) with a radius of 4 inches, according to the laws of physics, water will flow out of *A* and into *B* until the level (height) of water in both is the same.
   How many inches will the new height be?
   Express your answer as a decimal to the nearest tenth.
   [We ignore the volume of water in the connecting tube].



Answers

1. ________________
2. ________________
3. ________________

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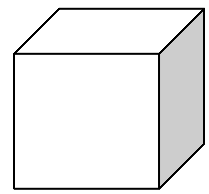
Solutions to Category 2 - Geometry

Meet #5, March 2010

Answers	
1.	24
2.	9.42
3.	1.8

1. A cube's volume is L^3 where L is its side's length. Therefore our cube is 2 feet long in each direction. Its surface is made of six squares measuring 2×2 feet each, or 4 square feet each, so the total is $6 \cdot 4 = 24$ square feet.



2. The ball's volume is $V = \frac{4}{3} \cdot \pi \cdot R^3 = \frac{4}{3} \cdot 3.14 \cdot 3^3 = 36 \cdot 3.14 = 113.04$ cubic inches. The pyramid's volume, which is the same, equals $\frac{1}{3} \cdot 6 \times 6 \cdot H$ where H is its height.

   We get that $H = \frac{3 \cdot V}{6 \cdot 6} = \frac{3 \cdot 113.04}{36} = 9.42$ inches.

3. The idea is to understand the the same volume of water – V – that initially fills cylinder A to a height of 5 inches, will be filling both A and B to a new height which we'll call H.

   A cylinder's volume is given by the formula $V = \pi R^2 H$ and so the volume of water to begin with is $V = \pi \cdot 3^2 \cdot 5 = 45 \cdot \pi$ cubic inches. This same volume will now be spread in both cylinders to a new height H, so we can write

   $V = 45 \cdot \pi = \pi \cdot H \cdot (3^2 + 4^2)$ and so we get $H = \frac{45}{3^2+4^2} = \frac{45}{25} = 1\frac{4}{5} = 1.8$ inches.

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Category 3 - Number Theory

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1. A subset of a set is a set containing any (or none) of the original set's elements, but not any that are not in the original set. A set can contain no elements at all.
   Set $A = \{1, 2, 3, 4, x\}$
   Set $B = \{1, 2, 3, 4, y\}$
   How many subsets do sets A and B have in common if $x \neq y$?
   (In other words, how many of A's subsets are also subsets of B?)

2. Sets A, B, and C all contain natural numbers that are $less\ than$ 30 according to the definitions below:
   $Set\ A = \{Multiples\ of\ 4\}$
   $Set\ B = \{Numbers\ that\ are\ 1\ less\ than\ a\ prime\}$
   $Set\ C = \{Multiples\ of\ 3\}$
   Find the sum of elements in the set $(A \cap C) \cup (B \cap C)$

3. In a class of 30 students, each student falls into at least one of these categories:
   Taller than 6 feet / *Vanilla lover* / *Great singer*.
   12 students love vanilla, 4 of whom are great singers.
   There are 2 great singers in the class who are taller than 6 feet, but only one of them loves vanilla. There are 14 great singers shorter than 6 feet who do not like vanilla.
   How many students are taller than 6 feet, dislike vanilla, and are not great singers?

Answers
1. _______________
2. _______________
3. _______________

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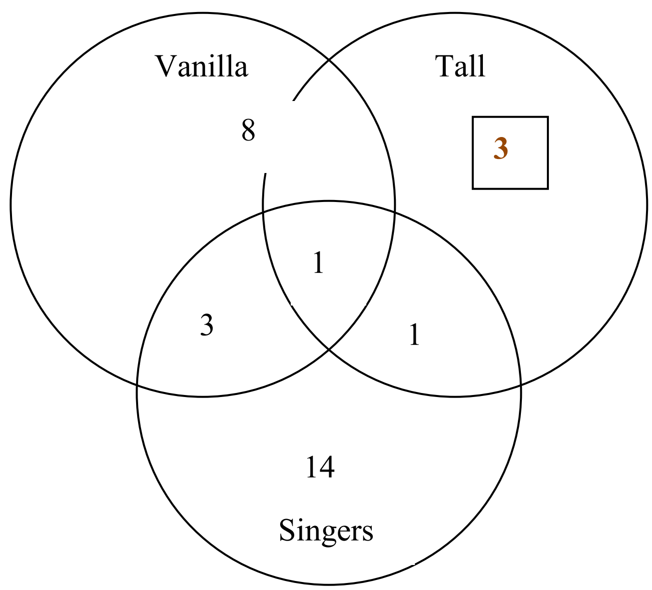
Solutions to Category 3 - Number Theory
Meet #5, March 2010

Answers	
1.	16
2.	60
3.	3

1. Both sets have 5 elements each, so each set has $2^5 = 32$ subsets.
   Since they differ in only one element ('x' in A vs. 'y' in B), we only need to exclude those subsets which include that element, or in other words, we can simply count all the subsets of the set $\{1, 2, 3, 4\}$. With 4 elements we have $2^4 = 16$ subsets :
   $\{\ \}, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\},$
   $\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}$
   You can make two observations from this –
   a. Each individual element of a set is part of exactly half of its subsets.
   b. The common subsets of A and B are the subsets of their intersection.

2. $A = \{4, 8, 12, 16, 20, 24, 28\}$
   $B = \{1, 2, 4, 6, 10, 12, 16, 18, 22, 28\}$
   $C = \{3, 6, 9, 12, 15, 18, 21, 24, 27\}$
   $A \bigcap C = \{12, 24\} \quad B \bigcap C = \{6, 12, 18\}$
   Therefore $(A \cap C) \cup (B \cap C) = \{6, 12, 18,24\}$ and the sum is 60.

3. We can count how many students are in each one of the sets or intersections.
   Though we can't tell how many - if any - of the 8 vanilla lovers who are not great singers are tall, it should be clear there are exactly 3 tall students in the group we seek (in order to add up to 30).
   No students are outside all of these sets.



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Category 4 - Arithmetic

Meet #5, March 2010

1. Of the 4 Marx brothers (Groucho, Chico, Harpo, and Zeppo), one will go to visit the zoo, one will visit the aquarium, one will visit grandpa, and one will stay home. How many different combinations of who-goes-where are there?
   To clarify – each destination is visited by a different person.

2. The Celtics basketball team has 12 players. If the coach selects 5 players at random as the starting five, what is the probability that both Paul Pierce and Kevin Garnett (two of the twelve players) were selected?
   Express your answer as a common fraction.

3. You have 3 coins. Two of them are fair, and the third has a 60% probability for showing Heads on each throw. When throwing the three coins together, what's the probability of getting **exactly** 2 heads?
   Express your answer as a decimal.

Answers
1. ________________
2. ________________
3. ________________

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Solutions to Category 4 - Arithmetic

Meet #5, March 2010

Answers	
1.	24
2.	5 / 33
3.	0.4

1. We can pick any one of the 4 brothers to go to the zoo, then one of the remaining 3 to go to the aquarium, then one of the last 2 to visit grandpa, and we are then left with the one that stays home. Since the choices are independent, we multiply the numbers: $4 \cdot 3 \cdot 2 \cdot 1 \equiv 4! = 24$

2. How many different combinations for the starting five exist? ${}_{12}C_5$.
   How many of these include both Garnett and Pierce? If we insist that these two players are included, then we only have to select 3 more players (out of the remaining 10), and we have ${}_{10}C_3$ ways of doing that.
   The probability then is $\frac{10C3}{12C5} = \frac{\frac{10!}{3!\times 7!}}{\frac{12!}{5!\times 7!}} = \frac{10!\times 5!}{12!\times 3!} = \frac{4\times 5}{11\times 12} = \frac{5}{33}$

3. Let's call the two fair coins #1 and #2, and the unfair coin #3. To get exactly two heads we need exactly one tail:
   $P\ (\#1 = T, \#2 = H, \#3 = H)\ =\ 0.5 \times 0.5 \times 0.6\ =\ 0.15$
   $P\ (\#1 = H, \#2 = T, \#3 = H)\ =\ 0.5 \times 0.5 \times 0.6\ =\ 0.15$
   $P\ (\#1 = H, \#2 = H, \#3 = T)\ =\ 0.5 \times 0.5 \times 0.4\ =\ 0.10$
   Added all up we get $P\ (2\ Heads, 1\ Tail)\ =\ 0.15 + 0.15 + 0.10\ =\ 0.4$
   Were all coins fair, the answer would be $3 \times 0.5^3 = 0.375 = \frac{3}{8}$.

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Category 5 - Algebra

Meet #5, March 2010

1. What is the positive difference between the two roots (solutions for *x*) of the equation below? [*Hint: both are integers*].

   $x^2 - 4 \cdot x = 21$

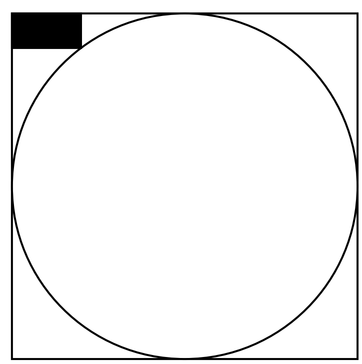
2. When each side of a square was increased in length by 50%, its area increased by 180 square inches.

   How many square inches are in the original square?

3. The diagram below shows a circle inscribed inside a square.

   The shaded rectangle measures 4×8 inches and touches the circle with one corner.

   How many inches are in the radius of the circle?



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5 - Algebra

Meet #5, March 2010

Answers	
1.	10
2.	144
3.	20

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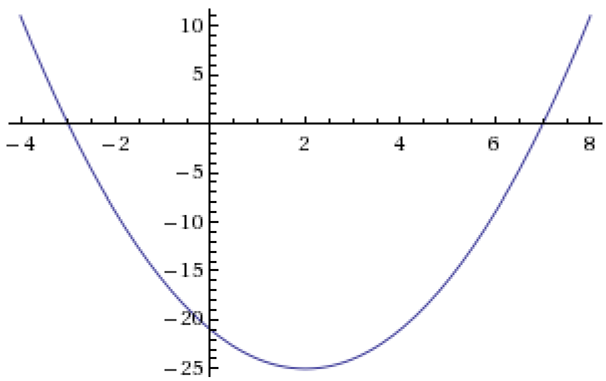
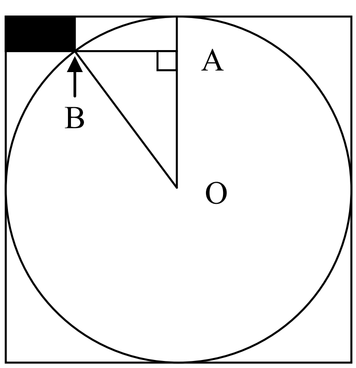
1. If we rearrange the equation $x^2 - 4 \cdot x = 21$ we can write it in the form: $(x + 3) \cdot (x - 7) = 0$ which makes clear that the solutions (roots) are $x = -3$ *and* $x = 7$ and the difference is 10. If you could not factor this way, you should have written $x^2 - 4 \cdot x - 21 = 0$ and then $x = \frac{4 \pm \sqrt{4^2 + 84}}{2} = 2 \pm 5$ to get the same values.

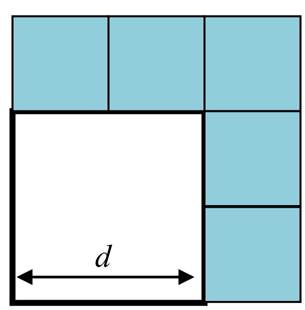


(x from −4 to 8)

The plot to the left shows the graph of $y = x^2 - 4 \cdot x - 21.$ This shows visually that x values of -3 and +7 result in $y = 0$ and are therefore the solutions.

2. If we call the original length d, then the original area is d^2. The increased length is $d \cdot 150\% = \frac{3}{2} \cdot d$ and the increased area is $(\frac{3}{2} \cdot d)^2 = \frac{9}{4} \cdot d^2$ so we know that the difference in areas is $\frac{5}{4} \cdot d^2 = 180$ and we get $d^2 = \frac{180 \cdot 4}{5} = 144$ square inches.





If the shaded area measures 180 square inches, then each little square is 36 square inches and so the original square is 4 times that area, or 144 square inches.

3. In triangle OAB we have $OB = R$ $(The\ circle's\ radius)$, $OA = R - 4$, $AB = R - 8$ and therefore: $(R - 4)^2 + (R - 8)^2 = R^2$, the solutions of which are $R = 4$ (invalid in our case) and $R = 20$.

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Category 6 - Team Questions

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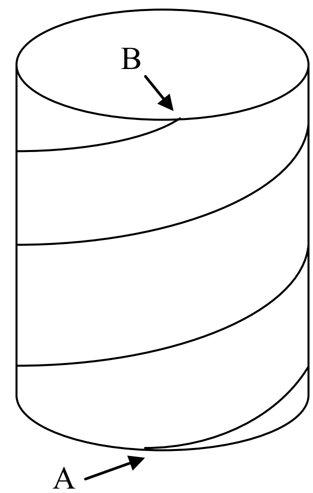

1. You painted half of a giant ball leaving 6,280,000 square inches unpainted. How many inches are in the ball's diameter? *Use* $\pi = 3.14$

2. The cylinder in the drawing has a radius of $\frac{6}{\pi}$ inches and a height of 15 inches. A string is wrapped around it in such a way that is has constant slope and completes 3 loops around the cylinder as shown, from point *A* to point *B* directly above it. How many inches long is the string?

3. Blue and Red balls are mixed in a box. The number of blue balls is B and the number of red balls is R. What is the probability that a randomly selected ball is blue if we know that $\frac{\text{B}-\text{R}}{\text{B}+\text{R}} = \frac{3}{4}$? *Express your answer as a common fraction.*

4. How many natural numbers up to 1,000 contain at least one of the digits '8' or '9'?

5. There are 6 states in New England (Maine, New Hampshire, Vermont, Massachusetts, Rhode Island, and Connecticut). You want to visit at least 3 on your upcoming trip. How many different combinations of states can you visit? *(The order in which you visit the states does not matter).*

6. Using the values you obtained in questions 1 through 5, evaluate the expression:

$$\frac{A*C+50}{\frac{D}{8}+E-2*B} * 10$$

Answers

1. ______________ = *A*
2. ______________ = *B*
3. ______________ = *C*
4. ______________ = *D*
5. ______________ = *E*
6. ______________

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Solutions to Category 6 - Team Questions

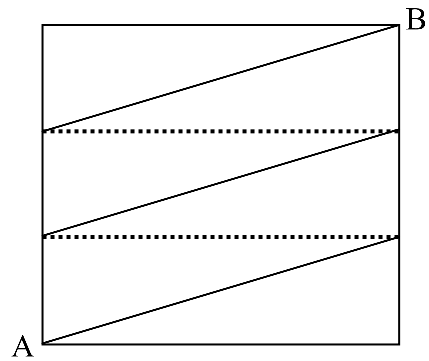
Meet #5, March 2010

Answers		
1.	2,000	$= A$
2.	39	$= B$
3.	7/8	$= C$
4.	488	$= D$
5.	42	$= E$
6.	720	

1. If R is the radius of the ball then its surface area is $4 \cdot \pi \cdot R^2$ and so one hemisphere's area is $A = 2 \cdot \pi \cdot R^2$. The Diameter then is

$$D = 2 \cdot R = 2 \cdot \sqrt{\frac{A}{2 \cdot \pi}} = 2 \cdot \sqrt{\frac{6{,}280{,}000}{6.28}} = 2 \cdot \sqrt{10^6} = 2{,}000\ inches$$

2. If we cut the cylinder along the line AB and flatten it, we get a rectangle measuring $2\pi R = 2\pi \cdot \frac{6}{\pi} = 12$ inches wide and 15 inches tall. The 3 loops of string become diagonals, each for a rectangle measuring 12 by 5 (a third of the height) inches. So each diagonal measures $\sqrt{5^2 + 12^2} = 13$ inches, and together the string measures 39 inches.



3. The probability that a randomly selected particle is blue, which we'll note as P(B), is defined as $\mathrm{P(B)} = \frac{\mathrm{B}}{\mathrm{B+R}}$, which means the relative share of Blue particles out of the total number $(\mathrm{B} + \mathrm{R})$.
We can rewrite $\frac{\mathrm{B-R}}{\mathrm{B+R}} = \frac{3}{4}$ as $4 \cdot (\mathrm{B} - \mathrm{R}) = 3 \cdot (\mathrm{B} + \mathrm{R})$ to conclude that $\mathrm{B} = 7 \cdot \mathrm{R}$ and so
$$\mathrm{P(B)} = \frac{\mathrm{B}}{\mathrm{B + R}} = \frac{7 \cdot \mathrm{R}}{7 \cdot \mathrm{R} + \mathrm{R}} = \frac{7}{8}$$
This measure, of the difference between two quantities divided by their sum, is called *Normalized Difference*, and ranges from -1 to $+1$, and is an alternative measure of concentration.

4. Listing all the possible scenarios is a little hard, but we can much more easily calculate the complement event – how many numbers do not contain either digit? There are 1,000 numbers overall ($= 10^3$ different combinations of 3 digits), and if we want to exclude the digits '8' and '9' then we limit ourselves to only 8 possible digits [1, 2, 3, 4, 5, 6, 7, 0]. With 8 digits we can have $8^3 = 512$ numbers, and so we have $1{,}000 - 512 = 488$ numbers that do contain at least one of these digits.
[With 3 digits we can write 1,000 numbers from 000 to 999. Whether you consider these 1,000 or the 1,000 numbers from 1 to 1,000 the answer is the same when we count numbers with the digits '8' or '9'].

5. The group of 6 states has 2^6 sub-groups (including the group itself and the empty group). Subtracting the empty group (no states visited), 6 sub-groups of only one state each, and ${}_2C_6 = \frac{6!}{2!\times4!} = \frac{5\times6}{2} = 15$ sub-groups of only 2 states, we're left with:

$2^6 - 1 - 6 - 15 = 64 - 22 = 42$ sub-groups of at least 3 states. (= ${}_3C_6 + {}_4C_6 + {}_5C_6 + {}_6C_6$)

6. $$\frac{A*C+50}{\frac{D}{8}+E-2*B} * 10 = \frac{2{,}000*\frac{7}{8}+50}{\frac{488}{8}+42-2*39} * 10 = \frac{1{,}750+50}{61+42-78} * 10 = \frac{1{,}800}{25} * 10 = 720$$

As you know, $720 = 6!$ and this number is part of two unique equalities:

$10! = 6! * 7!$ (Which is equivalent to noticing that $720 = 1*2*3*4*5*6 = 8*9*10$)

This is the only example of one factorial being the product of two other factorials (other than the trivial $6! = 5! * 3!$).

The other equality is $6! = 12^2 + 24^2$ the only known factorial that can be written as a sum of two squares.