You may use a calculator today!

Meet #4

February 2010

Intermediate

Mathematics League

of

Eastern Massachusetts

Meet #4

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Category 1 - Mystery

Meet #4, February 2010

1. Imagine all 7 billion people on Earth wanted to gather in one place. Let's assume we allow one square meter per person, and that we all gather inside a circle.
   How many kilometers are in the radius of that circle?
   Round your answer to the nearest whole kilometer.
   Use $\pi = 3.14$ *in your calculations.*

2. Looking at an analog clock, you can observe the minute-hand crossing over the hour-hand once in a while. How many seconds are in the interval between such crossings?
   Round your answer to the nearest second.

3. Hillary started saving \$50 a month in January 2004 (So she has \$50 at the end of January, \$100 at the end of February, and so on). In October 2004 Bill started saving \$70 a month. At the end of what month will Bill's balance be **greater** than Hillary's for the first time? *Express your answer in this format: MM/YYYY. For example January 2004 is written as 01/2004.*

Answers

1. ________________
2. ________________
3. ________________

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Solutions to Category 1 - Mystery

Meet #4, February 2010

Answers	
1.	47
2.	3,927
3.	08 / 2006

1. We want to find the radius of a circle with an area of 7 billion square meters;

   $$R = \sqrt{\frac{7 \cdot 10^9}{\pi}} \cong 47{,}215 \text{ meters} = 47.215 \text{ km} \cong 47 \text{ km}.$$

   This area (2,703 square miles) is just over a third of the size of Massachusetts (7,840 square miles). Surprised? (If you do the calculation for the U.S. population of 300 million, the answer is barely 10 kilometers).

2. Thinking of the 12-hour interval from 12:00am to 12:00pm we can count 11 crossings (the first a little after 1:05am, the second a little after 2:10am, and so on, and the last at exactly 12:00pm). So dividing 12 hours by 11 we get:

   $$\frac{12_{hours}}{11} = \frac{12 \times 60_{minu\ tes}}{11} = 65.\overline{45}_{minutes} = 65_{minutes}\ 27.\overline{27}_{seconds} \cong 3{,}927_{seconds}$$

3. Hillary's end-of-the month balance is $\$50 \cdot M$ where $M = 1$ is January 2004.
   Bill's balance is $\$70 \cdot N$ where $N = 1$ is October 2004, for which $M = 10$ (October 2004 is the 10^{th} month starting with January 2004), so we can write Bill's balance as $\$70 \cdot (M - 9)$, and so the requirement is $\$70 \cdot (M - 9) > \$50 \cdot M$ Or $20 \cdot M > 630$ or $M > 31.5$ so $M = 32$. The 32^{nd} month will be August 2006 written as 08/2006. ($32\ months = 2\ years\ +\ 8\ months$).
   Viewed a little differently: Bill saves an extra \$20 a month compared to Hillary. By the time he starts saving, she has already accumulated $\$50 \cdot 9 = \$450$, which will take him 22.5 (or 23) months to catch up with (at \$20 a month). $9 + 23 = 32$ as before.

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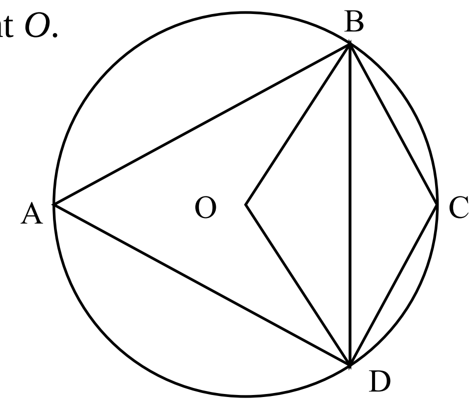
Category 2 - Geometry

Meet #4, February 2010

1. Assume that the Earth orbits the Sun along a perfect circular orbit with a radius of 150 million kilometers, and completes the orbit in 365 days and 6 hours.
   What is the Earth's average speed around the Sun?
   Express your answer in Kilometers per Hour (km/hour), rounded to the nearest integer. Use $\pi = 3.14$

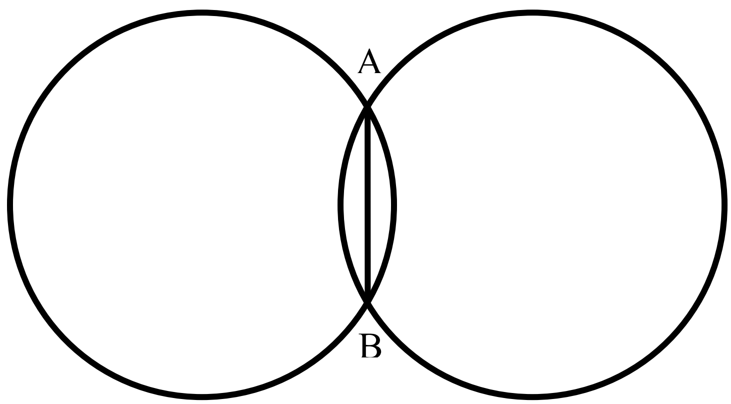
2. Kite *ABCD* is inscribed inside a circle whose center is point *O*.
   $\angle BDC$=25 degrees.
   How many degrees are in the measure of $\angle BOD$?
   [*A Kite is made up of two isosceles triangles*]



3. The radii (plural of radius) of both circles in the diagram measure 10 inches.
   They intersect each other in such a way that the distance *AB* measures 10 inches.
   How many inches are in the perimeter of the resulting shape?
   Express your answer in inches, rounded to the nearest hundredth. Use $\pi = 3.141592$.

Answers

1. ________________
2. ________________
3. ________________



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Solutions to Category 2 - Geometry

Meet #4, February 2010

	Answers
1.	107,461
2.	100
3.	104.72

1. Remember that Speed = Distance / Time.

   The distance, in kilometers, is the perimeter of the orbit, namely:

   $2 \cdot \pi \cdot R = 2 \cdot 3.14 \cdot 150 \cdot 10^6$ kilometers. The time, 365 days plus 6 hours, equals to $(365 \times 24 + 6)$ hours. The average speed then is:

   $$\frac{2 \cdot 3.14 \cdot 150 \cdot 10^6}{(365 \times 24 + 6)} = \frac{9.42 \cdot 10^8}{8{,}766} = 107{,}460.64 \ldots \cong 107{,}461\ km/hour$$

2. $\angle BDC = 25 = \angle DBC$ Therefore $\angle BCD = 130$ degrees (to complete to 180 degrees).

   $\angle BAD$ is inscribed on the chord *BD*, and so equals $180 - \angle BCD = 50$ degrees.

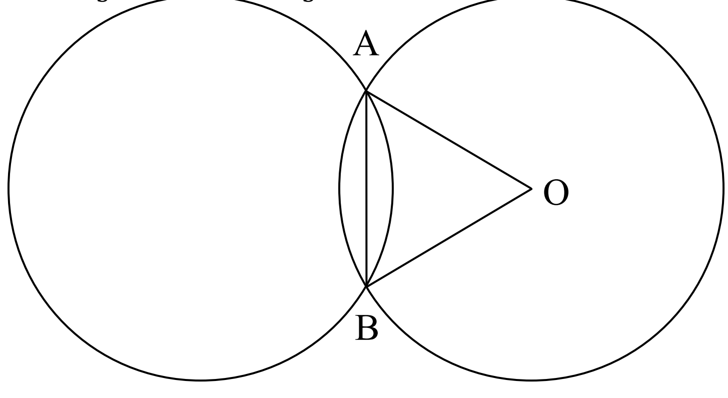
   Finally, $\angle BOD = 2 \times \angle BAD = 100$ degrees, since it's the central angle on the same chord. Something to think about: Does *ABCD* have to be a kite?

3. The perimeter is the sum of the two circles' perimeters, minus the two arcs $\overset{\frown}{AB}$.

   If we connect *A* and *B* to a circle's center *O*, we get an equilateral triangle, since we know that *AB* equals the radius of the circle. Therefore the central angle *AOB* measures 60 degrees, and so the arc $\overset{\frown}{AB}$ represents one-sixth of the circle's perimeter. This of course holds for the second circle as well, as it has the same radius.

   So the anwer is $\frac{5}{6}$ of the two perimeters, or

   $$\frac{5}{6} \times 2 \times 2 \cdot \pi \cdot R = \frac{10}{3} \cdot \pi \cdot 10 = \frac{100}{3} \cdot 3.141592 = 104.7197 \ldots \cong 104.72\ inches.$$



Category 3 - Number Theory
Meet #4, February 2010

1. What is the sum of the first two natural numbers that solve the equation below?

   $5 \times N \equiv 2\ (mod\ 11)$

   (*Reminder*: The Modulo 11 value of a number is the remainder when the number is divided by 11.)

2. *A* is the series $\{-30, -27, -24, -21, \ldots\}$

   B is the series $\{165, 163, 161, 159, \ldots\}$

   The N^{th} element of both series is the same. Find *N*.

   (In this question, $N = 1$ refers to the first element, $N = 2$ to the second element, etc.)

3. The numbers in the series $\{1, 2, 4, 8, 16, \ldots\}$ are colored in a Yellow/Red/Blue pattern:

   1 is painted Yellow, 2 Red, 4 Blue, 8 Yellow, 16 Red, etc.

   What is the color of the first number over one million?

Answers

1. _______________
2. _______________
3. _______________

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Solutions to Category 3 - Number Theory

Meet #4, February 2010

Answers	
1.	25
2.	40
3.	Blue

1. The meaning of $5 \times N \equiv 2\ (mod\ 11)$ in words is "if we multiply *N* by 5, and divide by 11, then the remainder is 2". To figure out what values of *N* will make that true we can simply try and see that $5 \times 7 = 35 = 33 + 2 \equiv 2\ (mod\ 11)$ and also $5 \times 18 = 90 = 88 + 2 \equiv 2\ (mod\ 11)$ so the solution is $7 + 18 = 25$.
   To directly solve we can write $N = (2 \div 5)(mod\ 11) = 2 \times 5^{-1}(mod\ 11)$ where 5^{-1} is the reciprocal of 5 in modulo 11 so that $5^{-1} \times 5 \equiv 1(mod\ 11)$. Writing down the multiplication table in modulo 11 we get that $5^{-1} \equiv 9(mod\ 11)$
   (As $5 \times 9 = 45 \equiv 1\ (mod\ 11)$).
   And indeed $2 \times 5^{-1} = 2 \times 9 = 18 \equiv 7(mod\ 11)$ is the solution. (And adding any multiple of 11 will also be a solution).

2. The N^{th} element of *A* is $a_N = (3 \cdot N - 33)$ while the N^{th} element of *B* is $b_N = (167 - 2 \cdot N)$. Equating the two: $3 \cdot N - 33 = 167 - 2 \cdot N$ we get $N = 40$.
   Check: $a_{40} = b_{40} = 87$.

3. The series is a geometric series where the N^{th} element is given by the formula $a_N = 2^{N-1}$ (Note that the first element is $1 = 2^0$). The first number over a million in the series is $2^{20} = 1{,}048{,}576$ which is the 21^{st} number.
   The colors follow a triple pattern, in which the 21^{st} color is the same as the 3^{rd} color, which in our case is Blue. ($21 = 3 = 0_{mod\ 3}$).

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Category 4 - Arithmetic

Meet #4, February 2010

1. Dora had $100 in her bank account.
   Her balance grew by 50%, then declined by 20%, then grew again by 10%.
   What is the final balance in Dora's account?

2. Every 100 pounds of ocean water consist of 3.5 pounds of salt and 96.5 pounds of water.
   A salt manufacturer filled a big pool with ocean water, let all the water evaporate, and ended up with 1,000 lbs of salt.
   How many pounds of water evaporated?
   Round your answer to the nearest integer.

3. The world record for the 100-meter dash is 9.58 seconds, held by Usain Bolt of Jamaica. The world record for a marathon (26.22 miles, 1 mile = 1,609 meters) is 2:03:59 (Two hours, three minutes, and 59 seconds) held by Haile Gebrselassie of Ethiopia.
   Comparing their average speeds, what fraction of Mr. Bolt's speed does Mr. Gebrselassie run?
   Express your answer as a percent, rounded to the nearest whole number percentage.

Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 4 - Arithmetic

Meet #4, February 2010

Answers	
1.	\$132
2.	27,571
3.	54%

1. $\$100 \times 150\% \times 80\% \times 110\% = \$150 \times 80\% \times 110\% = \$120 \times 110\% = \$132.$

2. The 1,000 lbs of salt is 3.5% of the initial weight (water + salt). So that initial weight is $\frac{1{,}000_{lbs}}{3.5\%} = \frac{1{,}000_{lbs}}{0.035} = 28{,}571.\overline{428571}_{lbs}$. The evaporated water then weighed that much minus the 1,000 lbs of salt, or $27{,}571.\overline{428571}_{lbs} \cong 27{,}571_{lbs}$.
[This of course is 96.5% of the combined weight].

3. In order to answer, we have to calculate and compare the runners' speeds, and in order to compare them we have to make sure we use the same basis for comparison. So calculating the runners' speeds in meters per second (m/s) we'd get:
Mr. Bolt's speed = $100\ meters\ /\ 9.58\ seconds\ =\ 10.438\ m/s$
Mr. Gebrselassie's speed =
$(26.22\ miles \times 1{,}609\ meters/mile)\ /\ (7{,}439\ seconds)\ =\ 5.671\ m/s$
[This is an average speed of about 4 minutes and 44 seconds per mile]
The percentage is $5.671\ /\ 10.438\ =\ 0.5433\ \cong\ 54\%$

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Category 5 - Algebra

Meet #4, February 2010

1. Dora and Diego had a lemonade selling competition.
   Diego was charging $0.25 a cup, and Dora decided to charge only $0.18 a cup at her stand. At the end of the day, it turned out that Dora sold 56 more cups than Diego did, but that both collected the same amount of money.
   How many cups did they sell altogether?

2. At Mike's favorite restaurant, a lunch made of one meat dish and two vegetarian side dishes costs $5.75, and a lunch made of 2 meat dishes and one vegetarian side dish costs $6.25.
   How much would a lunch made of 4 meat dishes and 7 vegetarian side dishes cost?
   Make sure to write your answer in the format $DD.CC ($Dollars.Cents).

3. The sum of two natural numbers is 3 times their difference, and their product equals 7,200.
   What is their sum?

Answers

1. ________________
2. ________________
3. ________________

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Solutions to Category 5- Algebra

Meet #4, February 2010

Answers	
1.	344
2.	$21.25
3.	180

1. If Diego sold *N* cups, then we know Dora sold $(N + 56)$ cups, and equating their earning (in cents) we can write:

   $25 \cdot N = 18 \cdot (N + 56)$ to get $7 \cdot N = 18 \cdot 56 = 1{,}008$ and so $N = 144$.

   Taken together, they sold $N + (N + 56) = 344$ cups. (Each collecting $36).

2. If we call the price of a meat dish *M*, and the price of a vegetarian dish *V*, then we know:

   $M + 2 \cdot V = \$5.75$ And $2 \cdot M + V = \$6.25$

   To solve, we can isolate *M* form the first equation: $M = \$5.75 - 2 \cdot V$ and substitute this in the second equation: $2 \cdot (\$5.75 - 2 \cdot V) + V = \$6.25$ to get

   $3 \cdot \mathrm{V} = \$5.25$ or $\mathrm{V} = \$1.75$ and so $\mathrm{M} = \$2.25$

   The answer therefore is $4 \cdot \mathrm{M} + 5 \cdot \mathrm{V} = 4 \cdot \$2.25 + 7 \cdot \$1.75 = \$21.25$

3. Calling our two numbers x, y we can write:

   $x \cdot y = 7{,}200$

   $x + y = 3 \cdot (x - y)$

   From the second equation we can get $x = 2 \cdot y$ and therefore $2 \cdot y^2 = 7{,}200$

   So $y = 60, x = 120$ and the sum $x + y = 180$.

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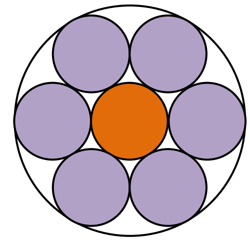
Category 6 - Team Questions
Meet #4, February 2010



1. The diameter of each of the shaded circles in the diagram to the right is 2 inches.
   How many square inches are in the total area of the 12 white spaces?
   Round your answer to the nearest hundredth.
   Use $\pi = 3.14$

2. A bucket is filled with 9 cups of wine, then the following takes place:
   One cup of water is poured in, the bucket is stirred, and then one cup is taken out of the bucket.
   This process is repeated 9 more times. After 10 times, how much of the bucket is still wine?
   Express your answer as a percent, rounded to the nearest whole number percentage.

3. The time it takes a passenger train to pass a freight train when both are traveling in the same direction is twice the time it takes when they travel in opposite directions.
   If the freight train travels at 10 mile per hour, how many miles per hour does the passenger train travel?

4. A series of circles is drawn such that the perimeter of the first circle is 1 inch, and the perimeter of each subsequent circle is 1 inch greater than the previous one.
   How many square inches are in the total area of the first ten circles in the series?
   Round to the nearest integer, and use $\pi = 3.14$

5. M and N are natural numbers (neither is 0) that solve: $(10 \cdot N)(mod\ 7) = (10 \cdot M)(mod\ 11)$.
   What is the least possible value for $M+N$?

6. Using the values you obtained in questions 1 through 5,
   evaluate the expression:
   $$\frac{E - \|A\|}{D - C} \times B \times 100$$
   Where $\|A\|$ is A rounded to the nearest integer.
   Note that B is a percentage.

Answers

1. _______________ = A
2. _______________ = B
3. _______________ = C
4. _______________ = D
5. _______________ = E
6. _______________

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Solutions to Category 6 - Team Questions

Meet #4, February 2010

Answers		
1.	6.28	= *A*
2.	35%	= *B*
3.	30	= *C*
4.	31	= *D*
5.	7	= *E*
6.	35	

1. The diameter of the bounding circle is 3 times the diameter of each of the small circles, or 6 inches. The white spaces are the difference between the area of the bounding circle and the total area of the 7 smaller circles:

   $White\ Area = A_{big\ circle} - 7 \cdot A_{small\ circle} = \pi \cdot R^2 - 7 \cdot \pi \cdot r^2$

   $= \pi \cdot (3^2 - 7 \cdot 1^2) = \pi \cdot 2 = 3.14 \cdot 2$

   $= 6.28\ square\ inches$

   (This is equivalent to the area of two small circles).

2. The key to see the solution is to understand that adding a cup of water dilutes the mixture, while removing a cup of the mixture does not change its concentration (only the total volume). Every time we add a cup of water, the total volume rises to 10 cups; 1 cup (10% of 10 cups) is purely water, and the other 9 cups are unchanged from the previous step. Seen differently, the portion of wine is diluted by a factor of 90%, since the same amount of wine that was previously part of a 9-cup bucket, is now spread over a 10-cup bucket. So mathematically speaking, the portion of wine after each step is 90% of what it was the step before – a geometric series.

   After 10 steps, the wine portion is $90\%^{10} = 34.86784 \ldots \% \cong 35\%$

3. The trains are not necessarily of the same length. Let's call the passenger train's length L_A and its speed V_A. Similarly we'll call the freight train's length L_B and its speed V_B. In order to pass, the passenger train has to advance a length of $L_A + L_B$ relative to the freight train. If both travel in the same direction, then the relative speed is $V_A - V_B$ and so the time it takes to pass is $T_1 = \frac{L_A+L_B}{V_A-V_B}$. If the trains travel in opposite directions the relative speed is $V_A + V_B$ and the passing time is $T_2 = \frac{L_A+L_B}{V_A+V_B}$. Therefore we know that $\frac{L_A+L_B}{V_A-V_B} = 2 \times \frac{L_A+L_B}{V_A+V_B}$. We can write this as: $\frac{1}{V_A-V_B} = \frac{2}{V_A+V_B}$ which simplifies to $V_A + V_B = 2 \times (V_A - V_B)$ and so $V_A = 3 \cdot V_B$.
   In our case, if the freight train's speed is 10 mph then the passenger train's speed is 30 mph.

4. The series of perimeters {P} is $\{1, 2, 3, 4, \dots 9, 10\}$ and so the series of radii {R} is $\left\{\frac{1}{2\pi}, \frac{2}{2\pi}, \dots, \frac{10}{2\pi}\right\}$, and the series of areas, $\{A = \pi R^2\}$ is $\left\{\pi \cdot \left(\frac{1}{2\pi}\right)^2, \pi \cdot \left(\frac{2}{2\pi}\right)^2, \pi \cdot \left(\frac{3}{2\pi}\right)^2, \dots, \pi \cdot \left(\frac{10}{2\pi}\right)^2\right\}$ which we need to sum.
   Note that although the perimeters and radii form simple arithmetic series, the areas do not, as we squared the radii. So we simply have to add these terms up, but it's useful to notice that we can take the common denominators out and write:
   $Sum\ of\ areas = \left(\frac{1}{2}\right)^2 \cdot \frac{1}{\pi} \times (1^2 + 2^2 + 3^2 + \cdots + 10^2) = \frac{1}{4} \cdot \frac{1}{3.14} \times 385 =$ $30.6528 \dots \cong 31$ square inches.

5. Since $(10 \cdot N)(mod\ 7)$ can only take on 7 possible values $(0 - 6)$, we can try different values for *N*, solve for *M*, and then find the pair with the minimal sum:

N	$10 \cdot N(mod\ 7)$	M	$M + N$
1	$10 \equiv 3$	8	9
2	$20 \equiv 6$	5	7
3	$30 \equiv 2$	9	12
4	$40 \equiv 5$	6	10
5	$50 \equiv 1$	10	15
6	$60 \equiv 4$	7	13
7	$70 \equiv 0$	11	18

As can be seen in the table, the minimal sum is $M + N = 7$.

6. $\frac{E - \|A\|}{D - C} \times B \times 100 = \frac{7 - \|6.28\|}{31 - 30} \times 35\% \times 100 = \frac{1}{1} \times 35 = 35$