Meet #3

January 2010

Intermediate

Mathematics League

of

Eastern Massachusetts

Meet #3

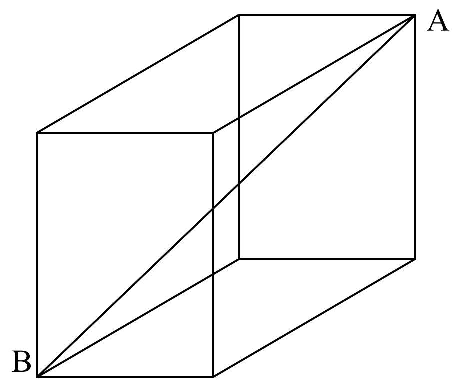
January 2010

Category 1 - Mystery

Meet #3, January 2010

1. Of all the number pairs whose sum equals their product, what is the minimal sum if at least one of the pair is a natural number?

2. The diagram below shows a rectangular prism measuring $3 \times 4 \times 12$ inches. What is the length of the big diagonal *AB*?



3. You have a bowl with candy bars. 3 friends visit you, one after the other, and to each you give half the bars in the bowl, plus two. After all visits you're left with only one bar. How many did you have to begin with?

Answers

1. ________________
2. ________________
3. ________________

Remember: You do not have to specify units. Specifying the wrong units will disqualify your answer.

Solutions to Category 1 - Mystery

Meet #3, January 2010

Answers	
1.	4
2.	13
3.	36

1. If we call our numbers x, y then the requirement is:

   $x + y = x \cdot y$ which we can write as $y = \frac{x}{x-1}$. If we substitute natural values for x, the resulting pairs are $\left\{(2,2), \left(3,\frac{3}{2}\right), \left(4,\frac{4}{3}\right), \left(5,\frac{5}{4}\right), \ldots etc.\right\}$ and the mimimal sum is of the first pair, 4. Note that for $x = 1$ we can't match another number.

   Lesser sums exist for numbers outside our range (for example for $\left(-1, \frac{1}{2}\right)$)

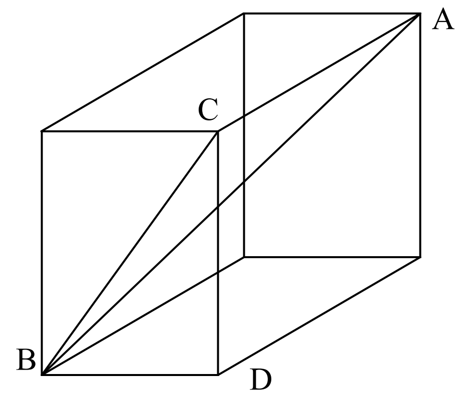
2. In right triangle BDC we have

   $BC^2 = BD^2 + DC^2 = 3^2 + 4^2 = 5^2$

   In right triangle BCA we have

   $AB^2 = BC^2 + CA^2 = 5^2 + 12^2 = 13^2$

   AB therefore measures 13 inches.



3. Working backwards from the end:

   If we call the number of bars in the bowl before any were given to the last friend $2 \cdot x$, then she was given $x + 2$ (one half plus 2), and then only one was left, so we need to solve: $(x + 2) + 1 = 2 \cdot x$ and the solution is $x = 3$. So there were $2 \cdot x = 6$ bars in the bowl just before the last visit.

   Similarly, we can write: $(y + 2) + 6 = 2 \cdot y$ to solve $2 \cdot y = 16$ bars before the second visit.

   Lastly, $(z + 2) + 16 = 2 \cdot z$ yields $2 \cdot z = 36$ candy bars to begin with.

Category 2 - Geometry

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1. The number of diagonals in a polygon is four times the number of its vertices.
   How many vertices does it have?
   (A diagonal is a line segment that connects two non-adjacent vertices of a polygon).

2. The exterior angle to a regular polygon N (with N sides) is half that of a regular polygon M (with M sides). Polygon N has 7 times as many diagonals as polygon M.
   What is the value of $M \cdot N$?

3. Tom stands exactly 2 miles west of Jerry.
   At 10:00am Tom starts walking east at 5 mph (miles per hour).
   At 10:20am Jerry starts heading north at 9 mph.
   How many miles between them at 12:00pm (noon)?

Answers

1. ________________
2. ________________
3. ________________

<u>Remember</u>: You do not have to specify units. Specifying the wrong units will disqualify your answer.

Solutions to Category 2 - Geometry

Meet #3, January 2010

Answers	
1.	11
2.	50
3.	17

1. If we call the number of vertices V, then the number of diagonals is given by the formula: $\frac{V\times(V-3)}{2}$ and for this to equal $4\times V$ we get $\frac{V-3}{2}=4$ or $V=11$.

2. The exterior angle is $\frac{360}{Number\ \ of\ sides}$ so it should be clear that $N=2M$.

   Polygon M will have $\frac{M\cdot(M-3)}{2}$ diagonals, and polygon N will have $\frac{N\cdot(N-3)}{2}=\frac{2M\cdot(2M-3)}{2}$ diagonals.

   So we need to solve: $2M\cdot(2\cdot M-3)=7\cdot M\cdot(M-3)$

   which we can simplify to $4\cdot M-6=7M-21$ and then to $3\cdot M=15$

   and the solution is $M=5,\ N=10$ and $M\cdot N=50$.

3. At 12:00pm, having walked for two hours, Tom is 10 miles to the east of his original location, which means he's 8 miles east of Jerry's original location.

   Jerry, after running at 9 mph for 1 hour and 40 minutes is 15 miles north of his (own) original location.

   The distance between them is $\sqrt{8^2+15^2}=\sqrt{289}=17\ miles.$

   Note that we have to compare their locations to some 'fixed' or agreed-upon point. You could have used any other point-of-reference and get the same answer.

Category 3 - Number Theory

Meet #3, January 2010

1. Evaluate the following expression. Write your result in scientific notation.

$$\frac{(2.3 \cdot 10^7) \cdot (4.8 \cdot 10^{-4})}{6 \cdot 10^5 + 9 \cdot 10^4}$$

2. Find x if x satisfies the equation:

$$1{,}071_{Base\ 10} = 3{,}060_{Base\ x}$$

3. Solve this Binary (base two) problem. Express your answer in Binary (base two).

$$11{,}111{,}111 - 101 \cdot 101{,}000 = ?$$

Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 3 - Number Theory

Meet #3, January 2010

Answers	
1.	1.6×10^{-2}
2.	7
3.	110,111

1. $\frac{(2.3\cdot10^{7})\cdot(4.8\cdot10^{-4})}{6\cdot10^{5}+9\cdot10^{4}} = \frac{2.3\cdot4.8\cdot10^{7-4}}{6.9\cdot10^{5}} = \frac{4.8\cdot10^{3}}{3\cdot10^{5}} =$
   $1.6\cdot10^{3-5} = 1.6\cdot10^{-2}$

2. The key is to realize that since in base *x* the number ends with a '0', then *x* is a factor of our number (1,071). That is true for any base (as an example, in base 10, numbers ending with a '0' are divisible by 10…). Factorization yields $1{,}071 = 3\times7\times51$, so obvious candidates are 3 and 7. [Other factors, like 51 or $3\times7 = 21$ would have been possible candidates, but since (in any base) $3{,}060 > 1{,}071$ it should be clear that $x < 10$]. So trying our 2 candidates: $1{,}071_{Base\ 10} = 110{,}200_{Base\ 3} = 3{,}060_{Base\ 7}$ we conclude that $x = 7$. Another way to think about it is to write $3x^3 + 6x = 1{,}071$ and to try some (obviously odd!) values for *x*, as we know $x < 10$.

3. We can do the Binary arithmetic.
First:

```
        101
  × 101,000
    101,000
   000,000
  101,000
 11,001,000
```

And then:

```
   11,111,111
 -  11,001,000
      110,111
```

Or we can translate everything to decimal:

$\mathbf{101} = 2^2 + 2^0 = 4 + 1 = 5$

$\mathbf{101,000} = 2^5 + 2^3$
$= 32 + 8$
$= 40$

$\mathbf{11,111,111}$
$= \mathbf{100,000,000 - 1}$
$= 2^8 - 1 = 256 - 1$
$= 255$

So our problem becomes $255 - 5\cdot40$
And the answer is
$55 = 32 + 16 + 4 + 2 + 1$
$= \mathbf{110,111}$

Category 4 - Arithmetic

Meet #3, January 2010

1. Evaluate the following expression:

$$\sqrt{(\sqrt{\sqrt{121}+5}+\sqrt[3]{216})^2-\ 2^6}$$

2. Evaluate the following expression:

$$\left(\frac{1}{2}\right)^{-1}\cdot\left(\frac{4}{3}\right)^{-2}\cdot(\sqrt[3]{125})^{-1}$$

Express your answer as a common fraction (A fraction of the form $^m/_n$ *which cannot be simplified).*

3. A perfect cube is a number of the form N^3 where N is a natural number. How many even perfect cubes that are less than one million exist?

Answers

1. ________________

2. ________________

3. ________________

Solutions to Category 4 - Arithmetic

Meet #3, January 2010

Answers	
1.	6
2.	9 / 40
3.	49

1. $\sqrt{\left(\sqrt{\sqrt{121}+5}+\sqrt[3]{216}\right)^2-\ 2^6}=\sqrt{\left(\sqrt{11+5}+6\right)^2-\ 64}=$

$\sqrt{(4+6)^2-64}=\sqrt{100-64}=\sqrt{36}=6$

2. $\left(\frac{1}{2}\right)^{-1}\cdot\left(\frac{4}{3}\right)^{-2}\cdot(\sqrt[3]{125})^{-1}=2\cdot\left(\frac{3}{4}\right)^2\cdot\frac{1}{\sqrt[3]{125}}=\frac{2\cdot 9}{16}\cdot\frac{1}{5}=\frac{9}{8\cdot 5}=\frac{9}{40}$

3. One Million $=10^6=100^3$

So there are 99 perfect cubes that are less than that, and 49 of them will be even.

(The cubes of all even numbers from 2 up to 98).

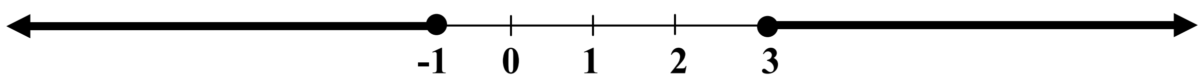
Category 5 - Algebra
Meet #3, January 2010

1. What is the positive difference between the least and largest integers that satisfy the inequality below?

$$\left|\frac{x}{5} - 1\right| \leq 2$$

2. What value of the parameter M in the inequality $M \cdot |x - 1| \geq 3$ will make the solution agree with the line graph below?

Express your answer as a decimal.



3. For how many integers N (*excluding zero*) does the inequality below hold true?

$$\left|\frac{30}{N}\right| > |2 \cdot N|$$

Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 5 - Algebra

Meet #3, January 2010

Answers	
1.	20
2.	1.5
3.	6

1. $\left|\frac{x}{5} - 1\right| \le 2$ can be written (multiply both sides by 5) as $|x - 5| \le 10$.

   If the argument on the left is positive we get $x - 5 \le 10\ so\ x \le 15$. If the argument is negative we get $x - 5 \ge -10\ so\ x \ge -5$. Taken together the range of values that make the inequality true are: $-5 \le x \le 15$.

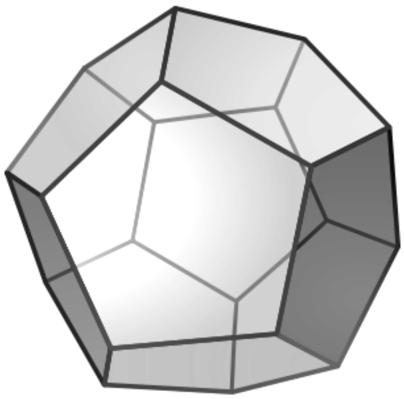
   The largest integer solution is 15, the least is -5, and the difference is therefore 20.

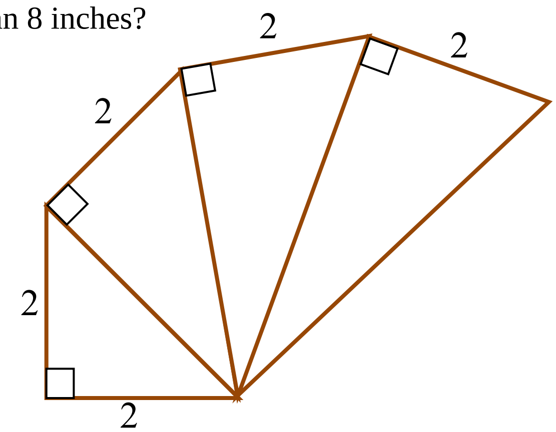
2. If the argument in the inequality $M \cdot |x - 1| \ge 3$ is positive then we can write:
   $M \cdot x - M \ge 3\ Or\ x \ge \frac{3+M}{M} = \frac{3}{M} + 1$ and if the argument is negative we can write:
   $M - M \cdot x \le 3\ Or\ x \le \frac{3-M}{-M} = \frac{M-3}{M} = 1 - \frac{3}{M}$
   Comparing these to the graph we require that $1 + \frac{3}{\text{M}} = 3$ AND $1 - \frac{3}{\text{M}} = -1$
   The solution to both is $\frac{3}{\text{M}} = 2$ Or $\text{M} = \frac{3}{2} = 1.5$.

   Alternatively, by looking at the graph solution we can ask ourselves: What inequality is described by the graph? And the answer is that the graph depicts points outside the region (-1, 3), or in other words points whose distance from the middle of the region is at least 2 (half the size of the region). So we can conclude that the graph depicts the inequality $|\text{x} - 1| \ge 2$ (1 being the midpoint of the region (-1, 3)).
   Comparing this inequality to the one given we can again calculate that $M = 1.5$ is the correct value.

3. Since both sides of the inequality are positive let's see what happens when *N* is positive: $\frac{30}{N} > 2 \cdot N$ means $N^2 < 15$ and so $N < 4$ or in other words *N* can take on the values $\{1, 2, 3\}$. If we assume *N* is negative, the same will apply (no need to reverse the inequality, as both sides are positive due to the absolute value), so *N* can take on the values $\{-1, -2, -3\}$. Overall there are 6 possible values for *N*.

1. One leg of each of the right triangles below measures 2 inches, and the other is the hypotenuse of the previous triangle. Including the triangles already drawn, how many such triangles do we have to draw in order to get a hypotenuse measuring more than 8 inches?





2. A regular Dodecahedron (shown above) is a regular polyhedron whose faces are 12 regular pentagons, such that three pentagons meet at each vertex. If we connect all vertices to each other with lines, some lines will be edges, some will be surface diagonals (diagonals of one of the face pentagons), and some will lie within the Dodecahedron's volume (called space diagonals).
How many space diagonals does the Dodecahedron have?

3. If A is the number of digits required to write 10^6 in Hexadecimal (base 16), and B the number of digits necessary to do so in Duodecimal (base 12), then find $A + B$.

4. We say that a polygon can be used as a tile if you can cover a big floor with a repeating pattern of tiles of that shape without leaving any part uncovered (and with no overlap). For example, we can use any rectangle (assume the floor goes on forever, and don't worry about its edges).
Find all regular polygons that can be used for tiling, and sum up the number of their sides.

5. What's the least natural number that satisfies the inequality below?

$$(N+1)^2 < \left(\frac{N}{2}\right)^3$$

6. Using the values you obtained in questions 1 through 5,

evaluate the expression: $\dfrac{E\cdot\sqrt{C\cdot D+1}}{\sqrt{A\cdot B}}$

Answers

1. ____________ = A
2. ____________ = B
3. ____________ = C
4. ____________ = D
5. ____________ = E
6. ____________

Solutions to Category 6

Team Questions

Meet #3, January 2010

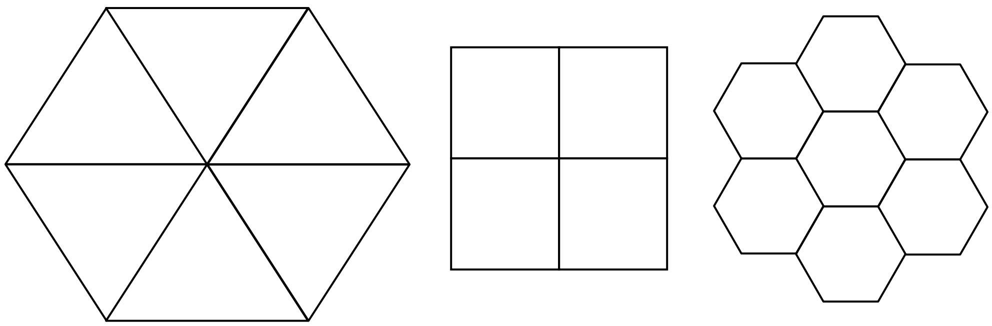
Answers		
1.	16	= *A*
2.	100	= *B*
3.	11	= *C*
4.	13	= *D*
5.	10	= *E*
6.	3	

1. Applying Pythagorean theorem, the first hypotenuse H_1 measures $\sqrt{2^2+2^2}=\sqrt{8}$ inches.
   Each consecutive hypotenuse will measure $H_{N+1}^2 = H_N^2 + 2^2$, so we get a series of lengths measuring: $\sqrt{8}, \sqrt{12}, \sqrt{16}, \sqrt{20}, \ldots$
   Remembering that $\sqrt{64}=8$, the first hypotenuse to be longer than that will measure $\sqrt{68}\ inches$, and in the series $\sqrt{8}, \sqrt{12}, \sqrt{16}, \sqrt{20}, \ldots$ this will be the 16th triangle.

2. There are 20 vertices and 30 edges [to see this, consider 12 pentagons = 12×5 = 60 vertices, but we have to divide by 3 since each vertex is shared by exactly 3 pentagons. Similarly, each edge is shared by exactly 2 pentagons, hence 30 edges]. From each vertex, 9 lines to nearby vertices will lie on a surface or along the edge (3 edges and 6 surface diagonals), so that leaves 10 diagonals going through the solid. 20 vertices × 10 inner diagonals from each = 200, but of course we'll have to divide by 2 as we counted each diagonal from both ends, so the final answer is 100.
   Another way to see this is to compute the total number of lines connecting vertices, which is $\frac{20\times 19}{2}=190$. From this, we have to subtract all the surface diagonals (12 pentagons, each having 5 diagonals, to a total of 60) and all 30 edges, again getting to 100 space diagonals.

3. In any base *x*, using *N* digits we can write x^N numbers (that includes zero), or in other words, the maximal value that can be written is (x^N-1). For example, in base 10, we need 7 digits to write 10^6, and only 6 to write $10^6-1=999{,}999$. So the question is to find the minimal values *A* and *B* such that $16^A > 10^6$ and $12^B > 10^6$. $16^4 = 65{,}536\ and\ 16^5 = 1{,}048{,}576\ \rightarrow A=5$.
   $12^5 = 248{,}832\ and\ 12^6 = 2{,}985{,}984\ \rightarrow B=6$ Therefore $A+B=11$.

4. If a regular polygon has N sides, then its exterior angle measures $\frac{360}{N}$ degrees and its interior angle measures $(180 - \frac{360}{N})$ degrees. If we are to use such a polygon for tiling, then at each vertex a number (M) of polygons has to meet in such a way that M interior angles will complete 360 degrees. In other words, the measure of the interior angle has to be a factor of 360. The only values for which this happens are $N = \{3,4,6\}$ (Which correspond to equilateral Triangles, Squares, and regular Hexagons). Added up we get $3 + 4 + 6 = 13$.



5. Although solving this inequality analytically is beyond middle-school level, you can use a little trial and error to figure this out. Your intuition should be that the right side of the inequality (third power) grows faster than the second-power left side. Trial and error should lead you to 10.

N	1	2	3	4	5	6	7	8	9	10
$(N+1)^2$	4	9	16	25	36	49	64	81	100	121
$\left(\frac{N}{2}\right)^3 = \frac{N^3}{8}$	$\frac{1}{8}$	1	$3\frac{3}{8}$	8	$15\frac{5}{8}$	27	$42\frac{7}{8}$	64	$91\frac{1}{8}$	125

You may also find it useful to rewrite the inequality as: $N^3 - 8 \cdot N^2 - 16 \cdot N - 8 > 0$ and then try out some values.

6. $\frac{E \cdot \sqrt{C \cdot D + 1}}{\sqrt{A \cdot B}} = \frac{10 \cdot \sqrt{11 \cdot 13 + 1}}{\sqrt{16 \cdot 100}} = \frac{10 \cdot \sqrt{144}}{\sqrt{1600}} = \frac{10 \cdot 12}{40} = \frac{120}{40} = 3$