Meet #2

December 2009

Intermediate

Mathematics League

of

Eastern Massachusetts

Meet #2

December 2009

Meet #2, December 2009
60 teams participated.
Average team score: 83. Median: 82

Average score by category: (to be posted here when scores are released)
1 Mystery: (out of 6)
2 Geometry:
3 Number Theory:
4 Arithmetic:
5 Algebra:
6 Team: (out of 36)

Category 1 - Mystery
Meet #2, December 2009

1. For Valentine's day, each student in Ms. Clavel's class sent a valentine to all other students. Overall, 600 valentines were sent. How many students are in the class?

2. What is the largest possible area *(in square inches)* for a rectangle with a perimeter of 120 inches?

3. When Alex bought his new cell phone, he could choose between two monthly plans:
   In plan A he had to pay 5 cents for every minute he'd talk on the phone.
   In plan B he had to pay $1 each month but only 3 cents for every minute of talk.
   He calculated that based on how much he talks, plan B would be $1.50 a month cheaper for him. How many minutes a month does he talk?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1 - Mystery

Meet #2, December 2009

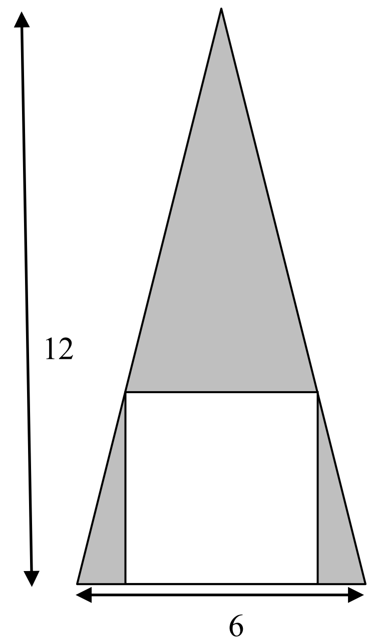
Answers	
1.	25
2.	900
3.	125

1. If there are N students in the class, then each one sends $(N-1)$ valentines, and overall there were $N \times (N-1)$ valentines sent.
   So we need to solve: $N \times (N-1) = 600$.
   Even if you're not sure yet how to solve this, a little trial and error with natural numbers can lead to $N = 25$.

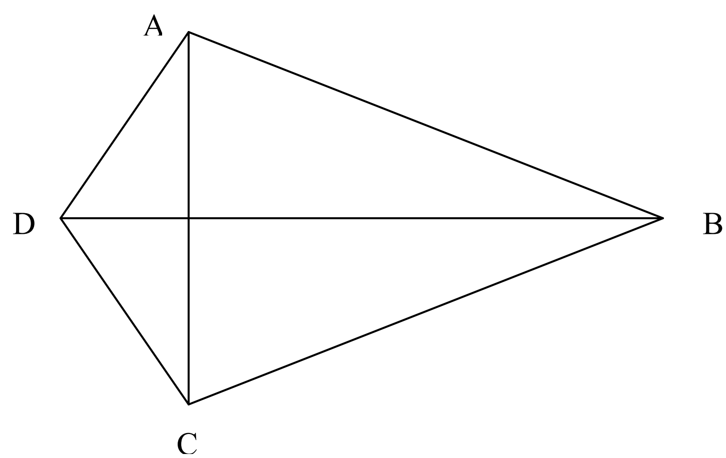
2. Though you may not be able to prove it yet, a little trial and error should point that of all rectangles of fixed perimeter, the square has the largest area. Stated differently, the product of two numbers whose sum is constant is largest when the numbers equal half that sum each. So in our case, a square with perimeter of 120 inches will have an area of $30^2 = 900 \; square \; inches$.

3. If Alex talks for T minutes, he'd have to pay $5 \cdot T$ cents in plan A, and $(3 \cdot T + 100)$ cents in plan B.
   Since plan B is cheaper, we can write: $5 \cdot T = (3 \cdot T + 100) + 150$ or $2 \cdot T = 250$, to get $T = 125$.
   Check: in plan A he'd pay \$6.25, and in plan B \$1+\$3.75=\$4.75

1. The big triangle's height is 12 units, its width is 6 units, and the inscribed square's perimeter is 16 units. How many square units are in the shaded area in the drawing?

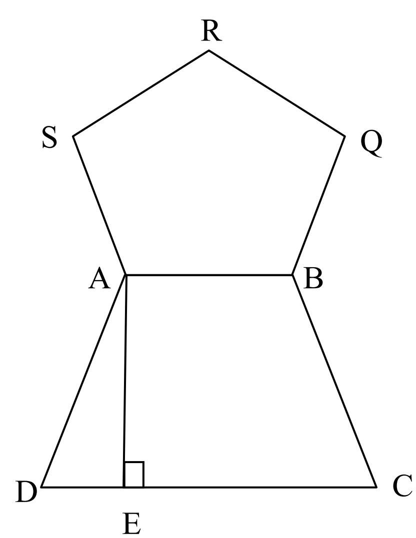


2. In the kite below, $AC = 18$ inches, $BD = 25$ inches. What is the length (in inches) of the perimeter of a square with the same area as the kite?



3. In the diagram below, the trapezoid $ABCD$ and the regular pentagon $ABQRS$ share a common edge AB.
$DC = 9$ inches.
$AE = 5$ inches.
Area of $ABCD = 40$ square inches.

What is the perimeter of ABQRS? *(in inches).*



Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 2 - Geometry

Meet #2, December 2009

Answers	
1.	20
2.	60
3.	35

1. The shaded area is the difference between the triangle's area and the square's area. The triangle's total area is $\frac{Width \times Height}{2} = \frac{6 \times 12}{2} = 36$ units squared. The square's area is $(\frac{16}{4})^2 = 16$ units squared, so the difference is 20 square units.
   Editor's note: The original problem was height 6, width 5, inscribed square's perimeter 12. Unfortunately, this square wouldn't fit in the triangle. The answers 6 and 7.5 were accepted.

2. As the kite is made up of two triangles, its area is $\frac{AC \times BD}{2} = \frac{18 \times 25}{2} = 225$ squared inches. A square with the same area will have a side of length $\sqrt{225} = 15$ inches, and therefore a perimeter of $15 \times 4 = 60$ inches.

3. Recall that the area of a trapezoid is $\frac{Height \times Sum\ of\ bases}{2}$ so in our case:
   $\frac{AE \times (AB + CD)}{2} = \frac{5 \times (9 + AB)}{2} = 40$ So we get $AB = 7$ inches.
   The perimeter of the pentagon is 5 times the length AB, or 35 inches.

Category 3 - Number Theory

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1. A and B are prime numbers that make this equation true:

   $49^2 \cdot 35 \cdot 8 = 5 \cdot 14^3 \cdot A^B$

   What is the value of $A + B$?

2. Ernie was counting the jellybeans in his bag.

   He noticed that when he arranged them in rows of either 6 or 8 beans, he was always left with 3 'extras', but when he arranged them in rows of 5, there were none left over. What's the least possible number of jellybeans that he has?

3. A floor measures 240 inches by 400 inches. You were asked to tile it with rectangular tiles so that a whole number of them would cover the floor (without a need to break any). You were also asked to use tiles which are twice as long as they are wide. You realize there are many different sizes you can use, but you want to minimize the number of tiles used. What's the least number of identical tiles you can use?

Answers

1. _______________
2. _______________
3. _______________

Solutions to Category 3 - Number Theory

Meet #2, December 2009

Answers	
1.	9
2.	75
3.	30

1. As we're dealing with natural numbers, all we have to do is write down the prime factorization of both sides of the equation and make sure all prime factors and their respective powers match each other:

   Left side: $49^2 \cdot 35 \cdot 8 = (7^2)^2 \cdot (5 \cdot 7) \cdot 2^3 = 2^3 \cdot 5 \cdot 7^5$

   Right side: $5 \cdot 14^3 \cdot A^B = 5 \cdot 2^3 \cdot 7^3 \cdot A^B$

   All that's missing on the right side is 7^2 So $A = 7$, $B = 2$ and $A + B = 9$.

2. The number of jelly beans can be divided by 5 (with no remainder), and leaves a remainder of 3 when divided by 8 and 6. It will therefore also leave a remainder of 3 when divided by any common multiple of 8 and 6, and specifically their *least common multiple*, which is 24. So we're looking for a number which is a multiple of 5, but is 3 more than a multiple of 24. If we list possible candidates: 27, 51, 75, 99,… we see that 75 is the first number to be a multiple of 5.

3. Since the tiles have to cover the floor exactly, one measurement has to be a factor of $400 = 2^4 \cdot 5^2$, and the other a factor of $240 = 2^4 \cdot 3 \cdot 5$. You can list all the factors, but from looking at the prime factorization we can conclude that the largest pair of factors with a ratio of $2:1$ has to be $(2^4 \cdot 5, 2^3 \cdot 5) = (80, 40)$. With this size tile we'll need 5×6 (or 10×3) $=$ 30 tiles to cover the floor.

Category 4 - Arithmetic

Meet #2, December 2009

1. Express $\frac{5}{16} + \frac{1}{9}$ as a decimal.

Use bar notation where appropriate.

2. Write $\frac{0.\overline{18}}{1-0.\overline{6}}$ as a simple fraction.

3. Mr. Bell spent a fifth of his money, then 12.5% of the remaining amount, then 10% of what was left, and finally a third of the remaining balance.

If he initially had $200, how much is left at the end?

Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 4 - Arithmetic

Meet #2, December 2009

Answers	
1.	$0.4236\overline{1}$
2.	6/11
3.	\$84

1. $\frac{5}{16}+\frac{1}{9}=0.3125+0.\overline{1}=0.4236\overline{1}$

2. If you recall that $\frac{1}{11}=0.\overline{09}$ then you'd realize that $\frac{2}{11}=0.\overline{18}$, if not you can write $x=0.\overline{18}$, $100x=18.\overline{18}$ to get $99x=18$ or $x=\frac{18}{99}=\frac{2}{11}$.

   In the denominator we have $1-0.\overline{6}=0.\overline{3}=\frac{1}{3}$ so overall we have $\frac{\frac{2}{11}}{\frac{1}{3}}=\frac{6}{11}$

3. We can follow this in steps:

   $\$200\cdot\frac{4}{5}=\$160$

   $\$160\cdot 87.5\%=\$160\cdot\frac{7}{8}=\$140$

   $\$140\cdot 90\%=\$126$

   $\$126\cdot\frac{2}{3}=\$84$

Category 5 - Algebra

Meet #2, December 2009

1. During a basketball game, your team scored three times as many 2-point field goals than it did 3-point field goals, and scored a total of 90 points.
   How many field goals did your team score?
   (There were no 1-point free-throws.)

2. The force by which an object in space is pulled by the Earth's gravity is proportional to $\frac{m}{R^2}$ where *m* is the mass of the object, and *R* its distance from the center of the Earth.
   Satellite #1 is orbiting Earth at a distance (from its center) of 5,200 miles.
   Satellite #2 is orbiting Earth at a distance of 20,800 miles, and has half the mass of satellite #1.
   What is the ratio of the gravitational force on satellite #1 to that on satellite #2?

3. The product of 3 consecutive even natural numbers divided by their sum is 64.
   What is the middle number?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5 - Algebra

Meet #2, December 2009

Answers	
1.	40
2.	32 or 32:1
3.	14

1. If we call the number of 3-point goals made G, then there were $(3 \cdot G)$ 2-point goals made, and the total score would be:

   $3_{points} \cdot G + 2_{points} \cdot (3 \cdot G) = 9 \cdot G = 90\, points.$

   So $G = 10$, and the total number of field goals made is $4 \cdot G = 40$ (Ten 3-pointers and thirty 2-pointers).

2. The ratio we seek is $\frac{m_1/{R_1}^2}{m_2/{R_2}^2}$, or $(\frac{R_2}{R_1})^2 \cdot \frac{m_1}{m_2}$ and we know that $\frac{m_1}{m_2} = 2\ and\ \frac{R_2}{R_1} = 4$

   so that the ratio is $4^2 \cdot 2 = 32$

3. If we call the middle number *x* then the problem is $\frac{(x-2)\cdot x\cdot(x+2)}{x-2+x+x+2} = 64$

   If we simplify this a bit we get $\frac{x\cdot(x^2-4)}{3\cdot x} = 64 = \frac{x^2-4}{3}$ Or $x^2 = 3 \cdot 64 + 4 = 196$

   Therefore $x = 14$. *($x = -14$ is a solution too, but we're looking for a natural number).*

Category 6 - Team Questions

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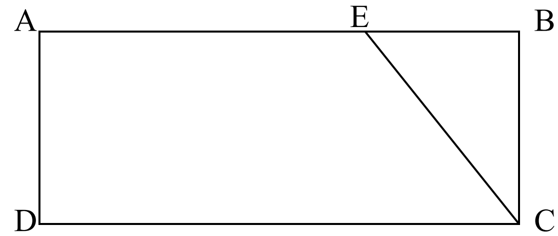


1. The perimeter of rectangle $ABCD$ is 140 inches. $\overline{EB}$ measures 4 inches less than $\overline{BC}$, and 1 inch less than half of $\overline{AE}$. What is the area, in square inches, of trapezoid $AECD$?

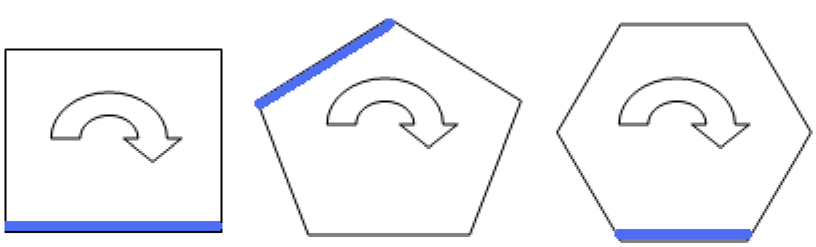
2. What is the sum of all natural numbers up to (and including) 100, that have an odd number of factors?

3. Pipe A can fill a pool in 10 hours, pipe B can fill the same pool in 5 hours, and pipe C can do it in 2 hours. If all pipes are pouring water together, how long will it take them to fill the pool? *Express you answer in minutes.*

4. A, B, and C are natural numbers such that the following is known:
   - The product $A \times B \times C$ is even
   - The number C is a power of 5
   - $GCF[A, B \times C] = 35$
   - $LCM[A, B, C] = 21{,}000$

   What is the least possible value of $A \times B$?

5. A square, a pentagon, and a hexagon are positioned as in the diagram below. Each shape has one specially-marked edge, as in the diagram. With each turn, each shape is turned clockwise onto its next edge. How many turns will it take until all special sides are pointing downward simultaneously?



6. Using the values you obtained in questions 1 through 5, evaluate the expression: $\left(\frac{B}{\left(\frac{D}{A}\right)} - E\right) \times C$

Answers

1. ______________ = A
2. ______________ = B
3. ______________ = C
4. ______________ = D
5. ______________ = E
6. ________________

Solutions to Category 6

Team Questions

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Answers		
1.	840	= *A*
2.	385	= *B*
3.	75	= *C*
4.	5,880	= *D*
5.	48	= *E*
6.	525	

1. If we call the length $\overline{EB} = x$, then we know that $\overline{BC} = x + 4$, and $\overline{AE} = 2 \cdot x + 2$.

   The overall perimeter is twice the sum of these three segments, so we can write:

   $2 \cdot (x + x + 4 + 2 \cdot x + 2) = 140$ from which we get $x = 16\ inches$.

   Therefore $\overline{BC} = 20\ inches$ and $\overline{AB} = 50\ inches$. The area of ABCD then is $20 \cdot 50 = 1{,}000\ square\ inches$, and the area of triangle EBC is $\frac{20 \cdot 16}{2} = 160\ square\ inches$.

   The trapezoid's area is the difference, namely $1{,}000 - 160 = 840\ square\ inches$.

2. For most numbers, we can pair their factors in pairs, for example:

   $30 = 1 \times 30 = 2 \times 15 = 3 \times 10 = 5 \times 6$. However, with numbers like

   $16 = 1 \times 16 = 2 \times 8 = 4^2$, one factor is multiplied by itself, which gives us an overall odd number of factors. So all the perfect squares (and only them) have an odd number of factors, and the answer is $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 + 100 = 385$.

   This can be seen more clearly if we write the prime factorization of any number:

   $N = {p_1}^{a_1} * {p_2}^{a_2} * {p_3}^{a_3} * \ldots .* {p_s}^{a_s}$ then we know that the number *N* has exactly $(a_1 + 1) \cdot (a_2 + 1) \cdot \ldots \cdot (a_s + 1)$ factors. If we want this number of factors to be *ODD* then all the arguments have to be odd (any even argument would make the whole product even), and so all the powers a_i have to be *EVEN*.

   That means we can write $N = {p_1}^{2b_1} * {p_2}^{2b_2} * {p_3}^{2b_3} * \ldots .* {p_s}^{2b_s} = M^2$, *N* is the square of some natural number *M*.

3. We have to think of how much work each pipe can do in one hour;

Pipe A fills $\frac{1}{10}$ of a pool, B can fill $\frac{1}{5}$, and C can fill ½.

If all work together and fill the whole (=1) pool in a period T hours then we have:

$(\frac{1}{2}+\frac{1}{5}+\frac{1}{10})\times T=1$ And we get $T = \frac{1}{(\frac{1}{2}+\frac{1}{5}+\frac{1}{10})} = \frac{1}{\frac{8}{10}} = \frac{10}{8} = 1\frac{1}{4}$ hour = 75 minutes.

4. Since $A \times B \times C$ is even, at least one of the numbers (A, B, C) is even.

Since *C* is a pure power of 5, *C* is odd, and so at least one of (A, B) is even.

$GCF[A, B \times C] = 35 = 5 \cdot 7$ means that 5^1 (and no higher power) is a factor of *A* (maybe of *B*, but not necessarily), and that 7 is a factor of both *A* and *B* (either *A* or *B* could have a higher power of 7, but not both). This also means that only one of the numbers (A, B) is even, otherwise the GCF would have been even.

$LCM[A, B, C] = 21{,}000 = 21 \times 1{,}000 = 3 \cdot 7 \cdot 2^3 \cdot 5^3$ means that whichever is the even number, it has 2^3 as a factor, and also that 3 is a factor of either *A* or *B* (but not both, otherwise $GCF[A, B \times C]$ would have had 3 as a factor).

Regarding 7, we now conclude that neither *A* nor *B* may have a higher power of 7.

So we know that both *A* and *B* contain a 7, *A* contains a 5, one number contains 8, and one number contains 3. The only uncertainty is whether *B* contains a 5 (up to the third power), but since we're interested in the least possible value for the product, we'll assume it does not.

When multiplied, we'll get $8 \cdot 3 \cdot 5 \cdot 7^2 = 5{,}880$. No other factors can be present in either *A* or *B*.

Editor's note: In one cluster, this was the most difficult problem on the team round.

5. For the square and the hexagon, we need to complete a whole number of revolutions together, so the answer is the least common multiple of 4 and 6, which is 12. Every 12 turns, these two shapes will point downward together again. For the pentagon, we need a whole number of revolutions, plus three turns, so 3, 8, 13, 18, 23… turns.

Taken together, we're looking for a multiple of 12 that is 3 more than a multiple of 5;

12, 24, 36, 48, 60, … 48 is the first such multiple.

6. $\left(\frac{B}{\frac{D}{A}} - E\right) \times C = \left(\frac{385}{\frac{5880}{840}} - 48\right) \times 75 = \left(\frac{385}{7} - 48\right) \times 75 = (55 - 48) \times 75 = 7 \times 75 = 525$