Meet #1

October 2009

Intermediate

Mathematics League

of

Eastern Massachusetts

Meet #1

October 2009

www.imlem.org

Meet #1, October 2009
61 teams participated.
Average team score: 104. Median: 110

Average score by category:

1 Mystery:	2.7 (out of 6)
2 Geometry:	3.0
3 Number Theory:	2.5
4 Arithmetic:	3.9
5 Algebra:	5.0
6 Team:	12.5 (out of 36)

Of 588 Regulars, 34 students received perfect scores of 18 points.
Only 2 teams got a perfect Team Round score: Lexington Clarke and Newton Oak Hill.
Lexington Clarke got 214 points in this meet, missing just one question in Mystery.

Category 1 - Mystery

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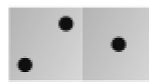
1. A Domino tile has 2 numbers on it, each being one of the numbers [0, 1, 2, 3, 4, 5, 6]. A set of 'double-six' domino tiles contains exactly one tile of each 2-number combination. How many tiles are in the set?

This is an example of a [2, 1] tile -



2. Container A contains 990 milliliters of water and 10 milliliters of wine. Container B contains 1,995 milliliters of water and 5 milliliters of wine. The contents of both containers are poured into a big bowl and thoroughly mixed. 200 milliliters of liquid are then taken out in a cup – how much wine (in milliliters) ends up in the cup?

3. What natural number is 5 times the sum of its digits?

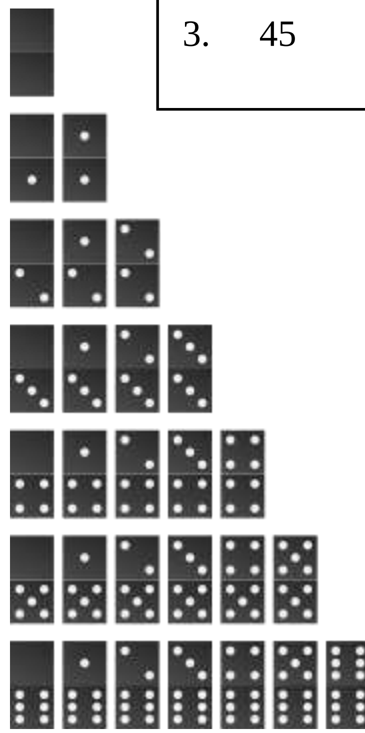
Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1 - Mystery

Meet #1, October 2009

Answers	
1.	28
2.	1
3.	45

1. We can see this in two ways: The number '6' appears on 7 tiles ([6, 0], [6, 1], [6, 2], [6, 3], [6, 4], [6, 5], [6, 6]), similarly the number '5' appears on 7 tiles, but we already counted the [6, 5] tile, so only need to add 6 tiles (those with '5', but without '6'), then 5 more tiles with '4' (but without '5' or'6') etc. The answer is $7 + 6 + 5 + 4 + 3 + 2 + 1 = 28$ tiles.
We could also see it this way: Each number appears on 8 half tiles (count the '6's in the list above).
So we have 7 numbers × 8 half tiles = 28 tiles.



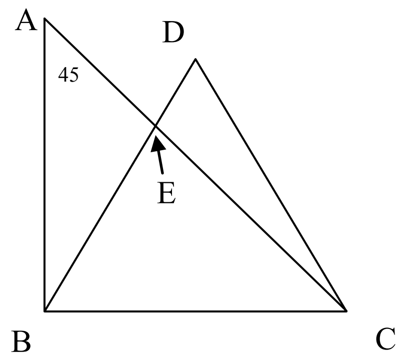
2. Taken together, we have 3,000 milliliters (3 liters) of liquid – 15 milliliters wine and 2,985 milliliters water. The relative share of wine is $\frac{15}{3,000} = \frac{1}{200} = 0.5\ percent.$
The same share is expected to be found in the cup: 0.005×200 milliliters = 1 milliliter.

3. The sought after number clearly has to have more than 1 digit, so let's start our search for a matching 2-digit number. If our 2-digit number is written as XY then its value is $(10 \cdot X + Y)$, and so the requirement is that $(10 \cdot X + Y) = 5 \cdot (X + Y)$, which we can simplify to write $5 \cdot X = 4 \cdot Y$.
Remember that X and Y are single digits, and the only digits to match this requirement are $X = 4$ and $Y = 5$, and indeed $\mathbf{45 = 5 \cdot (4 + 5)}$.
Could there be a 3-digit number XYZ that matches the requirement so that:
$100 \cdot X + 10 \cdot Y + Z = 5 \cdot (X + Y + Z)$?
NO: The maximum value possible for the right side is $5 \cdot (9 + 9 + 9) = 135$ requires $X = 1$, but the maximal value with $X = 1$ is $5 \cdot (1 + 9 + 9) = 95$ which is not a 3-digit number.

Category 2 - Geometry

Meet #1, October 2009

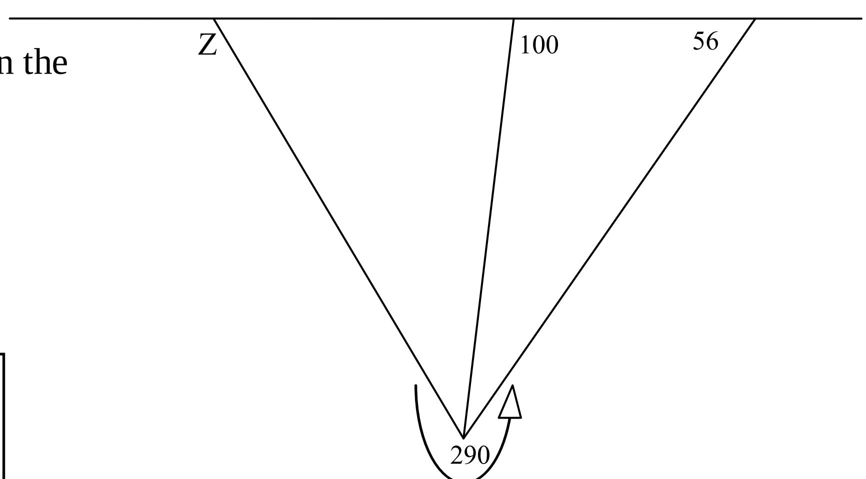
1.



In the above diagram, *right triangle* ABC and *equilateral triangle* BCD intersect at point E. Given the angle measurements above, how many degrees in the measure of ∠BEC?

2. The supplement to an angle x is two-and-a-half times its complement. How many degrees in the measure of x?

3. How many degrees in the measure of angle Z?



Answers	
1.	________________
2.	________________
3.	________________

Solutions to Category 2 - Geometry

Meet #1, October 2009

Answers	
1.	75
2.	30
3.	126

1. ∠CBD=60 degrees, therefore ∠DBA=30 degrees, therefore ∠BEA =105 degrees, and so ∠BEC equals 75 degrees.

   Another way to see this: ∠ EBC=60 degrees, ∠BCE=45 degrees, therefore ∠CEB=75 degrees.

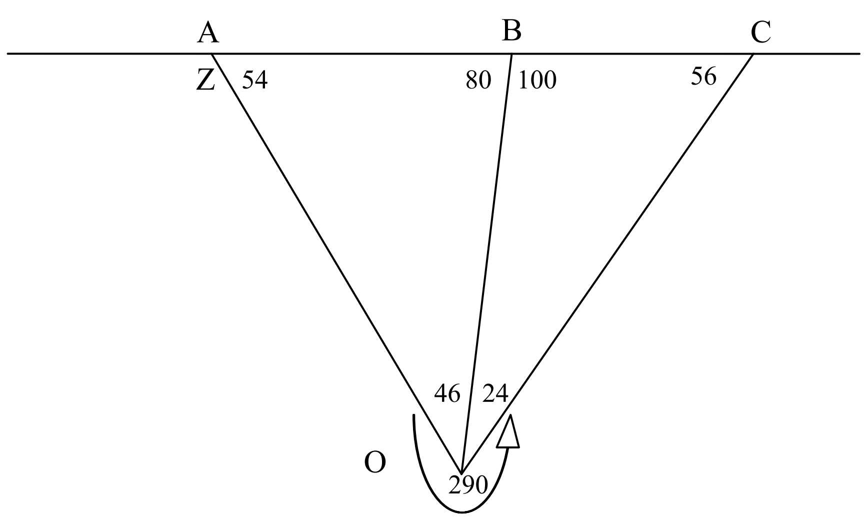
2. We can write the equation $(180 - x) = 2.5 \cdot (90 - x)$ and solve: $1.5 \cdot x = 0.5 \cdot 90$ and so $x = 30$ degrees.

3. By completing triangle CBO to 180 degrees we get ∠BOC=24 degrees.

   By completing around O to 360 degrees we get ∠BOA=46 degrees.

   Complete triangle ACO to 180 to get ∠OAC=54 degrees.

   Finally Z supplements it, so is 126 degrees.



Category 3 - Number Theory

Meet #1, October 2009

1. What is the positive difference between the sum of the 6 largest primes that are less than 20 and the sum of the 3 smallest composites that are greater than 30?

2. The number $1X38X$ is divisible by 12 when $X = A$, and by 9 when $X = B$. What is the value of $A+B$?

3. If S_{66} represents the sum of all the factors of the number 66 and S_{70} represents the sum of all the factors of the number 70, then find the value of $S_{70} - S_{66}$.

Answers

1. _______________
2. _______________
3. _______________

Solutions to Category 3 - Number Theory

Meet #1, October 2009

Answers	
1.	27
2.	3
3.	0

1. The first primes are 2, 3, 5, 7, 11, 13, 17, 19,… so the sum of the 6 largest under 20 is $5 + 7 + 11 + 13 + 17 + 19 = 72$.

   The first composites greater than 30 are 32, 33, 34 and their sum is 99.

   The positive difference is $99 - 72 = 27$.

2. $1A38A$ has to be divisible by 12 and so both by 3 and by 4.

   Divisibility by 3 means that the sum of digits $(2 \cdot A + 12)$ is a multiple of 3, which means A can be one of the digits [0, 3, 6, 9]. Divisibility by 4 means that the number 8A has to be divisible by 4, so A can be one of the digits [0, 4, 8]. The only digit to match both criteria is $A = 0$.

   $1B38B$ is divisible by 9, so its sum of digits has to be divisible by 9. That means that $(2 \cdot B + 12)$ is a multiple of 9, and the only possible value for the digit B is 3.

   Therefore $A + B = 0 + 3 = 3$.

3. $S_{66} = 1 + 2 + 3 + 6 + 11 + 22 + 33 + 66 = 144$

   $S_{70} = 1 + 2 + 5 + 7 + 10 + 14 + 35 + 70 = 144$

   $S_{70} - S_{66} = 0$

Category 4 - Arithmetic

Meet #1, October 2009

1. Find the value of the expression below:

   $5 \cdot (4^2 + 4)^2 + 45 \div 15 \cdot 3$

2. The table below lists individual points scored by players of the Orlando Magic in game 3 of this year's NBA finals.

   What is the positive difference between the *mean* individual score and the *median* individual score?

H. TURKOGLU	18
R. LEWIS	21
D. HOWARD	21
C. LEE	4
R. ALSTON	20
M. PIETRUS	18
T. BATTIE	4
J. NELSON	2
M. GORTAT	0

3. Express $\dfrac{\frac{2\cdot3\cdot5\cdot6}{6\cdot6}}{\frac{6+6}{\frac{6}{6}}}$ as a decimal rounded to the nearest hundredth.

Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 4 - Arithmetic

Meet #1, October 2009

Answers	
1.	2,009
2.	6
3.	0.42

1. $5 \cdot (4^2 + 4)^2 + 45 \div 15 \cdot 3 = 5 \cdot (16 + 4)^2 + 3 \cdot 3 =$
   $5 \cdot 20^2 + 9 = 5 \cdot 400 + 9 = 2{,}009$

2. If we list the scores from least to greatest {0, 2, 4, 4, **18**, 18, 20, 21, 21}, we can see that the ***median*** score is 18.

   Added up, there were 108 points scored, so the ***mean*** is $\frac{108}{9} = 12$.

   The difference is $18 - 12 = 6$.

3. $\frac{\frac{2 \cdot 3 \cdot 5 \cdot 6}{6 \cdot 6}}{\frac{6+6}{\frac{6}{6}}} = \frac{\frac{180}{36}}{\frac{12}{1}} = \frac{5}{12} = 0.41\bar{6} \cong 0.42$

Category 5 - Algebra

Meet #1, October 2009

1. Given these values:

   $A = \frac{1}{3}, B = 5, C = 1, D = \frac{2}{3}, E = \frac{3}{2}, F = -1$

   What is the value of the expression: $A \cdot \frac{B+C}{D \cdot E} + F$?

2. Solve for x: (What is the value of x that makes this true?)

   $3 \cdot (x - 5) + 40 = (-2) \cdot x + 5$

3. For what value of M will the solution of the following equation be $x = \frac{M}{2}$?

   $2 \cdot M + x = 3 \cdot (M - 1)$

Answers

1. __________________
2. __________________
3. __________________

Answers	
1.	1
2.	-4
3.	6

Solutions to Category 5

Algebra

Meet #1, October 2009

1. $A \cdot \frac{B+C}{D \cdot E} + F = \frac{1}{3} \cdot \frac{5+1}{\frac{2}{3} \cdot \frac{3}{2}} + (-1) = \frac{1}{3} \cdot \frac{6}{1} - 1 = 2 - 1 = 1$

2. $3 \cdot (x-5) + 40 = (-2) \cdot x + 5$

   $3x - 15 + 40 = -2x + 5$

   $5x + 25 = 5$

   $5x = -20$

   $x = -4$

3. If we know that $x = \frac{M}{2}$ is the solution, we just have to put it back in the original equation and solve for M:

   $2M + x = 2M + \frac{M}{2} = 3M - 3$ which is simplified to $\frac{M}{2} = 3$ or M=6.

   We could have also simplified the original equation to be $x = M - 3$ and then sub in $x = \frac{M}{2}$ to get the same result.

Category 6 - Team Questions

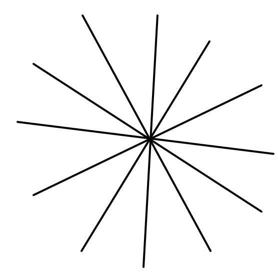
Meet #1, October 2009

1. List A contains all the prime numbers that are less than 50.
   List B contain all the natural numbers less than 50 that are one more than a positive multiple of 6.
   What is the arithmetic mean value of the numbers the two lists have in common?

2. The distance from Providence to New-York is 180 miles.
   A train rode the first half of the distance at a speed of 60 miles-per-hour (mph), and the second half at 90 mph. What was the train's average speed?

3. There are two distinct pairs of natural numbers A and B that satisfy the equation:
   $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=1-\frac{1}{A}-\frac{1}{B}$ What is the positive difference between the sum of values of the first pair $(A+B)$ and the sum of values of the second pair?

4. Bart drew M line segments in such a way that all intersect at their midpoints creating equal angles between them. Lisa did the same with N line segments. They noticed that the difference between the measures of their respective angles was 27 degrees. What is the least possible value of $M+N$?



5. 30 students are taking a test where the highest possible score is 100.
   At least 10 students will score 90 or higher and exactly 10 students will score 60 or lower.
   What is the difference between the highest and lowest possible *mean* scores on the test?
   Round your scores to the nearest integer before calculating the difference.

6. Using the values you obtained in questions 1 through 5, evaluate the expression below:
   $\frac{B}{C+\frac{3\cdot D}{2}}\cdot A-E$

Answers

1. _______________ $= A$
2. _______________ $= B$
3. _______________ $= C$
4. _______________ $= D$
5. _______________ $= E$
6. _________________

Solutions to Category 6 - Team Questions

Meet #1, October 2009

Answers	
1.	25 = *A*
2.	72 = *B*
3.	39 = *C*
4.	14 = *D*
5.	37 = *E*
6.	-7

Number of teams getting each question correct, out of 61 teams:

Q1	Q2	Q3	Q4	Q5	Q6
48	25	25	7	19	3

1. List A={2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}

   List B={7, 13, 19, 25, 31, 37, 43, 49}

   The numbers in common are {7, 13, 19, 31, 37, 43}

   The mean value is $\frac{7+13+19+31+37+43}{6} = \frac{150}{6} = 25$

2. A wrong answer is to average 60 and 90 to get 75 mph.

   The correct way is to recall that the definition of average speed is total distance divided by total time. The total distance is of course 180 miles, but how long did it take the train to travel it? The first leg took $\frac{90\text{miles}}{60\text{mph}} = 1\frac{1}{2}$ hour. The second leg took $\frac{90\text{miles}}{90\text{mph}} = 1$ hour. The total time then is $2\frac{1}{2}\,hours$.

   The average speed then is $\frac{180\text{ miles}}{2\frac{1}{2}\text{hours}} = \frac{180}{\frac{5}{2}} = \frac{2*180}{5} = 72$ mph.

3. $\frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{20+15+12}{60} = \frac{47}{60} = 1 - \frac{1}{A} - \frac{1}{B} = \frac{60}{60} - \left(\frac{1}{A} + \frac{1}{B}\right)$

   Therefore $\left(\frac{1}{A} + \frac{1}{B}\right) = \frac{13}{60}$ and we're looking for natural numbers A, B to satisfy this.

   A natural guess is to try $\frac{1}{A} = \frac{10}{60}$ *and* $\frac{1}{B} = \frac{3}{60}$ that leads to $A = 6, B = 20$ and $A + B = 26$. The other pair that solves this is $A = 5, B = 60$ that gives $A + B = 65$.

   The difference then is $65 - 26 = 39$.

   No other natural solutions exist, since A and B have to be factors of 60.

4. For Bart, there are M intersecting segments, and therefore $2 \cdot M$ angles at the intersection, so each one measures $\frac{360}{2 \cdot M} = \frac{180}{M}$ degrees.

   Similarly, Lisa's angles measure $\frac{180}{N}$ degrees each, and we know that:

   $\frac{180}{M} - \frac{180}{N} = 27$ degrees [for our purposes, it does not matter which is bigger, since we're only interested in the sum $M + N$, so we can assume $M < N$].

   Though this is a little 'scary' we know that M and N are integers, so we can substitute a few values;

M	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
180/M	90	60	45	36	30	25.7	22.5	20	18

   We can notice that a 27 degrees difference occurs between values of $M = 4$ and $N = 10$, so $M + N = 14$.

   [Other natural solutions are $(N = 60, M = 6), (N = 20, M = 5)$ but we were looking for the least value].

5. The highest mean score requires that the lowest 10 students score 60, and everyone else scores 100, in which case the mean is $\frac{10\times60+20\times100}{30} = 86.\overline{6} \cong 87$

   The lowest score requires that the lowest 10 score 0, the top 10 score 90, and the middle 10 score 61 (so still exactly 10 students score '60 or lower').

   In this case the mean score is $\frac{10\times0+10\times61+10\times90}{30} = 50.\overline{3} \cong 50$

   The difference is $87 - 50 = 37$.

6. $\frac{B}{C+\frac{3\cdot D}{2}} \cdot A - E = \frac{72}{39+\frac{3\cdot 14}{2}} \cdot 25 - 37 = \frac{72}{39+21} \cdot 25 - 37 = \frac{72}{60} \cdot 25 - 37 = \frac{6}{5} \cdot 25 - 37 = 30 - 37 = -7$