You may use a calculator today.

Meet #5
March 2009

Intermediate Mathematics League of Eastern Massachusetts

Meet #5
March 2009

Category 1
Mystery
Meet #5, March 2009

1. Sam told Mike to pick any number, then double it, then add 5 to the new value, then divide the new value by 3, then subtract 2 from the new number, then multiply that new number by 8 and lastly subtract 27 from that new value. Sam asked Mike what his final number was and Mike told him it was 13. What was the original number Mike picked?

2. What is the 50th smallest positive integer that has a remainder of 3 when divided by 8?

3. From January 1st, 2008 to July 1st, 2008, the value of Mark's stock decreased $37\frac{1}{2}$%. From July 1st, 2008 to January 1st, 2009 the value of Mark's stock returned to the value it had on January 1st, 2008. By what percent did the value of Mark's stock rise from July 1st, 2008 to January 1st, 2009?

Answers

1. _______________
2. _______________
3. _______________

Solutions to Category 1
Mystery
Meet #5, March 2009

Answers

1. 8

2. 395

3. 60

1. Working backwards using inverse operations we can find Mike's number.

$$13 + 27 \rightarrow 40 \div 8 \rightarrow 5 + 2 \rightarrow 7 \times 3 \rightarrow 21 - 5 \rightarrow 16 \div 2 \rightarrow 8$$

2. The smallest positive integer with a remainder of 3 when divided by 8 is 3. The next one would be $3 + 8 = 11$, followed by 11 + 8 = 19, and so on. So the n^{th} number would be $3 + (n - 1)(8)$ and the 50^{th} number would be 3 + 49(8) = 395.

3. A decrease of $37\frac{1}{2}\%$ is a decrease of $\frac{3}{8}$ down to $\frac{5}{8}$ of the original. To get back to the original price it would have to increase by $\frac{3}{8}$ of the original. Increasing the current $\frac{5}{8}$ by $\frac{3}{8}$ gives us $\frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5} = 60\%$.

Another way to look at this is that since the July value is $\frac{5}{8}$ of the original value, the original value is $\frac{8}{5}$ of the July value which is $1\frac{3}{5}$ or 160%. That's a 60% increase from the July value.

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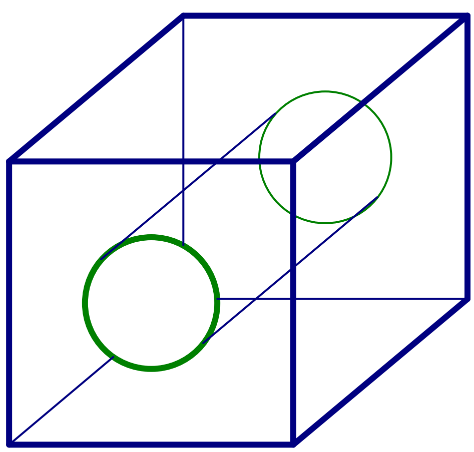
Category 2
Geometry
Meet #5, March 2009

1. What is the total surface area of a solid hemisphere with radius 12 cm? Use 3.14 as an estimation for π and express your answer as a decimal.

2. The radius of cone A is 6 times as long as the radius of cone B. The height of cone A is one-ninth the height of cone B. If the volume of cone A is 200 cm^3, how many cm are in the volume of cone B?

3. In the diagram below, an 8 inch by 8 inch by 8 inch cube has a cylinder with radius 2 in drilled through the center of one face all the way through the cube and out the opposite face. What is the surface area of the resulting figure? Use 3.14 as an estimation for π and express your answer as a decimal.



Answers

1. _______________
2. _______________
3. _______________

Solutions to Category 2
Geometry
Meet #5, March 2009

Answers

1. 1356.48

2. 50

3. 459.36

1. The surface area of a sphere is found by the formula $\text{SA} = 4\pi r^2$, so a hemisphere would have a surface area of $2\pi r^2$. However, there is also a circular base to a hemisphere which has an area of πr^2, for a total surface area of $3\pi r^2$. Since the radius is 12 cm, the surface area is $\text{SA} = 3\pi(12)^2 = 3\pi(144) = 432\pi \approx 1356.48$

2. Volume of a cone is found by the formula $V = \frac{1}{3}\pi r^2 h$. If r is the radius of cone B, then $6r$ is the radius of cone A. If h is the height of cone B, then $\frac{h}{9}$ is the height of cone A. We can then write a formula for the volume of cone A as $V_A = \frac{1}{3}\pi(6r)^2\left(\frac{h}{9}\right)$, while the volume of cone B is $V_B = \frac{1}{3}\pi r^2 h$. Since we know the volume of cone A is 200, we can substitute 200 for V_A and simplify to get:
$V_A = \frac{1}{3}\pi(6r)^2\left(\frac{h}{9}\right) \rightarrow 200 = \frac{1}{3}\pi 36r^2\left(\frac{h}{9}\right) \rightarrow 200 = \frac{4}{3}\pi r^2 h \rightarrow 50 = \frac{1}{3}\pi r^2 h = V_B$

3. Describing the surface area in words, there are 6 squares, although 2 of them have circular holes cut out of two of them. There is also "tube" through the middle of the cube which is a cylinder without bases. So the surface area of the whole thing is:
$6(8)^2 - 2\pi(2)^2 + 2\pi(2)(8) =$
$384 - 8\pi + 32\pi =$
$384 + 24\pi \approx$
$384 + 24(3.14) =$
$384 + 75.36 =$
459.36

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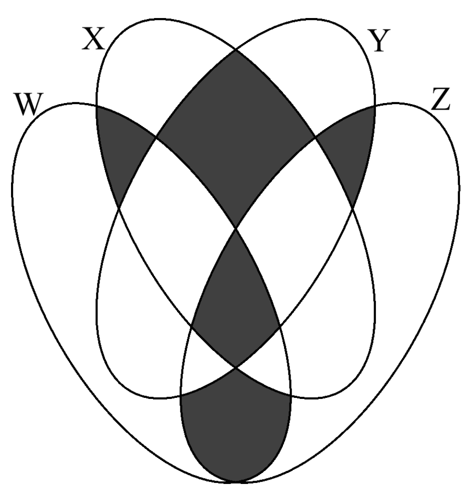
Category 3
Number Theory
Meet #5, March 2009

1. At the summer math camp there are 50 kids taking math classes. Thirty-five kids took the Number Theory class and 24 kids took the Probability class. If 7 kids took neither of the two classes, how many kids took both classes?

2. The set A contains only the vowels a, e, i, o, u and y. How many subsets of set A contain the letter y?

3. In the Venn diagram below, the four ovals represent 4 sets of numbers as described below. How many numbers fall in the shaded regions?

W = the set of positive even integers less than 50
X = the set of prime numbers less than 50
Y = the positive multiples of 3 less than 50
Z = the perfect squares less than 50



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 3
Number Theory
Meet #5, March 2009

Answers

1. 16

2. 32

3. 5

1. 50 – 7 = 43 kids took at least one of the two classes. A nice way to think about this is that there are 35 Number Theory books being used and 24 Probability books being used for a total of 59 books. If each of the 43 kids had one book there would be 16 left over books. So 16 of the kids must take a second book and are taking both classes.

2. Subsets of set A can contain anywhere from none of the elements of A up to containing all of the elements of A. Since we are looking for the subsets that contain "y", we just need to determine whether or not the other letters are in a subset. There are 5 other letters and each can either be in the subset or not be in the subset resulting in 2 possibilities for each letter. That makes for $2 \times 2 \times 2 \times 2 \times 2 = 32$ possible subsets.

3. The five shaded areas represent the overlap between X and Y, between Y and Z, between Z and W, between X and W, and the region where all 4 overlap.

X and Y contain just 3
Y and Z contain just 9
Z and W contain 4 and 16
X and W contain just 2
The region where all circles overlap would be empty

That's a total of 5 numbers in the shaded regions.

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Category 4
Arithmetic
Meet #5, March 2009

1. The coach at the Instant Messaging Legion of Extraordinary Mathematicians had 13 students tryout for the IMLEM team. The coach must choose 10 of them to compete at the upcoming meet. How many different groups of 10 students can the coach choose from the 13 students?

2. Bugsy was at the penny candy store where they have two big barrels of mixed candy. One of the barrels contains candy which costs 3, 5, 6, 7, 8 or 13 cents each. The other barrel contains candy which costs 2, 4, 5, 6, 8 or 11 cents. If Bugsy chooses 1 piece of candy from each barrel, what is the probability that the total cost is an even number of cents? Express your answer as a common fraction.

3. How many different 5-digit zip codes are possible if the only restriction is that no two consecutive digits can be the same?

Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 4
Arithmetic
Meet #5, March 2009

Answers

1. 286

2. $\frac{4}{9}$

3. 65610

1. Choosing 10 of the 13 students is the same as choosing 3 of the students to not be on the team. Choosing 3 of 13 can be found as a combination by: ${}_{13}C_3 = \frac{13!}{(13-3)!\cdot 3!} = \frac{13\cdot 12\cdot 11}{3\cdot 2\cdot 1} = 286$ ways to choose 3 students not on the team and 10 on the team.

2. The table below shows the possible sums of the costs of the two pieces of candy. There are 36 possible ways to pick the two pieces of candy and 16 of them have even total costs. That's $\frac{16}{36} = \frac{4}{9}$.

+	3	5	6	7	8	13
2	5	7	8	9	10	15
4	7	9	10	11	12	17
5	8	10	11	12	13	18
6	9	11	12	13	14	19
8	11	13	14	15	16	21
11	14	16	17	18	19	24

Alternatively we can use case work to figure this one out. In order for the sum of two whole numbers to be even, either both numbers are even or both are odd. If both numbers are even there is a $\frac{2}{6}$ chance the candy from the first barrel has an even price and a $\frac{4}{6}$ chance the candy in the second barrel has an even price, so the probability of both candies having an even price is $\frac{2}{6} \times \frac{4}{6} = \frac{8}{36}$. Similarly, the probability that both have odd number prices is $\frac{4}{6} \times \frac{2}{6} = \frac{8}{36}$. The probability of either of those happening is $\frac{8}{36} + \frac{8}{36} = \frac{16}{36} = \frac{4}{9}$.

3. The first digit of the zip code could be any of the 10 digits. The second digit could not be the same as the first digit so there are 9 choices. The third digit cannot be the same as the second, but it could be the same as the first, so 9 choices again. Similarly there are 9 choices for the 4th and 5th digits. That's a total of $10 \times 9 \times 9 \times 9 \times 9 = 65610$ choices.

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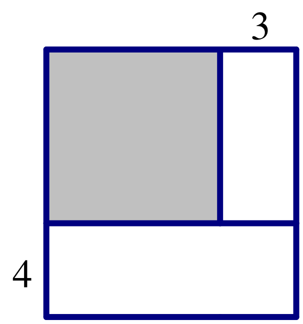
Category 5
Algebra
Meet #5, March 2009

1. What is the positive difference between the two solutions to the equation below?

$$3x^2 + 12x - 31 = 32$$

2. The diagram below is a large rectangle which is made up of a square and 2 rectangles as shown. The area of the entire figure is 756. What is the area of the square?



3. The difference between a positive number and $3\frac{1}{3}$ times its reciprocal is equal to $1\frac{1}{6}$. What is the number? Express your answer as a common fraction.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5
Algebra
Meet #5, March 2009

Answers

1. 10

2. 576

3. $\frac{5}{2}$

1. $3x^2 + 12x - 31 = 32$

$3x^2 + 12x - 63 = 0$

Dividing both sides of the equation by 3 gives us:

$x^2 + 4x - 21 = 0$

$(x + 7)(x - 3) = 0$

$x + 7 = 0$ or $x - 3 = 0$

$x = -7$ or $x = 3$

The positive difference between two solutions is $3 - (-7) = 10$

2. If we call the side of the square x, the square has area x^2, the rectangle to the right of the square has area $3x$ and the area of the rectangle below the square is $4(x + 3) = 4x + 12$.

The total area of the three shapes is $x^2 + 3x + 4x + 12 = x^2 + 7x + 12 = 756$.

$x^2 + 7x - 744 = 0$

$(x + 31)(x - 24) = 0$

$x + 31 = 0$ or $x - 24 = 0$

$x = -31$ or $x = 24$

Since the side length of the square cannot be negative, the side of the square is 24 and the area of the square is $24^2 = 576$.

3. $x - 3\frac{1}{3}\left(\frac{1}{x}\right) = 1\frac{1}{6}$

$x - \frac{10}{3}\left(\frac{1}{x}\right) = \frac{7}{6}$

$x - \frac{10}{3x} = \frac{7}{6}$

Multiplying both sides of the equation by $6x$ gives us:

$6x^2 - 20 = 7x$

$6x^2 - 7x - 20 = 0$

$(2x - 5)(3x + 4) = 0$

$2x - 5 = 0$ or $3x + 4 = 0$

$2x = 5$ or $3x = -4$

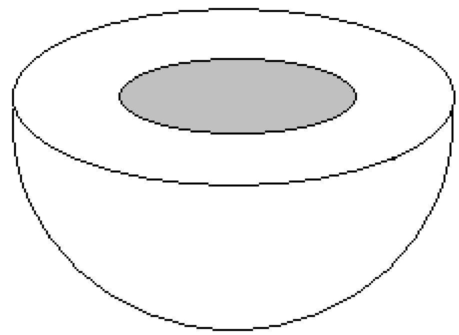
$x = \frac{5}{2}$ or $x = -\frac{4}{3}$

Since we know x is positive, it must be $\frac{5}{2}$.

Category 6
Team Questions
Meet #5, March 2009

1. Sam flipped a coin five times and wrote either H for 'heads' or T for 'tails' after the result of each flip onto a single piece of paper. How many different sequences of five letters could Sam end up writing?

2. In the diagram on the right, a hemisphere with diameter 14 inches has a second hemisphere with radius 4 inches carved out of it. What is the surface area of the resulting figure? Use 3.14 as an estimation for π and round your answer to the nearest whole number.



3. Twelve students are at the park waiting to play basketball. Only 10 can play at a time. The park referee chooses which ten will play and then those ten are split into a "blue" team and a "red team", each with 5 players. How many different ways can the two teams turn out?

4. A set which contains x distinct elements has 99 different subsets with at least 3 elements. What is the value of x?

5. There are 4 rational solutions to the equation $n^4 - 13n^2 + 36 = 0$. What is the sum of the absolute values of those solutions?

Answers

1. _______________ = *A*
2. _______________ = *B*
3. _______________ = *C*
4. _______________ = *D*
5. _______________ = *E*
6. _________________

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression:

$$\frac{\dfrac{C}{(D-1)^2} \div \sqrt[4]{\dfrac{B}{A}} \times E}{3}$$

Solutions to Category 6
Team Questions
Meet #5, March 2009

Answers

1. 32

2. 512

3. 16632

4. 7

5. 10

6. 770

1. Since there are two equally likely choices for each flip of the coin there are 2^5 possible outcomes for the 5 flips. $2^5 = 32$.

2. The surface area of a sphere is $4\pi r^2$, so the surface area of the outer half of this shape would be $2\pi r^2$ or $2\pi(7)^2 = 98\pi$. The inner "bowl" shape would have surface area $2\pi(4)^2 = 32\pi$. The top of the bowl is basically a large circle with a small circle cut out, the area of which is $\pi(7)^2 - \pi(4)^2 = 49\pi - 16\pi = 33\pi$. The combined surface area of the entire shape is $98\pi + 32\pi + 33\pi = 163\pi \approx 163(3.14) = 511.82 \approx 512$.

3. There are several ways to do this problem. You could choose 2 kids to sit out. That's ${}_{12}C_2 = \frac{12\cdot 11}{2\cdot 1} = 66$ and then choose 5 of the remaining 10 to be on the blue team (the rest would automatically be on the red team). That's ${}_{10}C_5 = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 252$. Together that's $66 \times 252 = 16632$ ways to pick teams.

There are many similar ways to do this problem, another of which would be to choose 5 of the 12 to be on the red team and then choose 5 of the remaining 7 to be on the blue team. That's ${}_{12}C_5 = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 792$ and then ${}_7C_5 = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 21$ with a total of $792 \times 21 = 16632$ ways again.

4. In a set with x elements, the total number of subsets is always 2^x since there are two choices for each element (to be in the subset or not). There is always 1 subset which is empty, and x subsets with exactly one element. There are also ${}_xC_2 = \frac{x(x-1)}{2}$ sets with 2 elements. That means that $2^x - \frac{x(x-1)}{2} - x - 1 = 99$. Looking at the closest power of 2 to 99, we have $2^7 = 128$. So, if there are 7 elements, there are 128 total subsets, 1 empty set, 7 subsets with 1 element and $\frac{7(6)}{2} = 21$ subsets with 2 elements. The number of subsets with at least 3 subsets is 128 – 1 – 7 – 21 = 99, so x must equal 7.

5. At first glance, this problem could appear rather hard. However if we replace n^2 in the equation with just x we would get the new equation $x^2 - 13x + 36 = 0$ which factors into $(x - 9)(x - 4) = 0$ and so $x = 9$ or $x = 4$. Remember though that we are looking for n, not x. Since $n^2 = x$, either $n^2 = 9$ or $n^2 = 4$. Making n either $3, -3, 2,$ or -2. The sum of the absolute values of those solutions is:
$|3| + |-3| + |2| + |-2| = 3 + 3 + 2 + 2 = 10$

6.

$$\frac{\frac{C}{(D-1)^2} \div \sqrt[4]{\frac{B}{A}} \times E}{3} = \frac{\frac{16632}{(7-1)^2} \div \sqrt[4]{\frac{512}{32}} \times 10}{3} =$$

$$\frac{\frac{16632}{36} \div \sqrt[4]{16} \times 10}{3} = \frac{462 \div 2 \times 10}{3} = \frac{2310}{3} = 770$$