Intermediate
Mathematics League
of
Eastern Massachusetts

Category 1
Mystery
Meet #4, February 2009

1. How many integers between 1 and 100 are 5 more than a positive multiple of 6?

2. What would the 45th letter be in the pattern below?

A B B C C C D E E F F F G H H I I I ...

3. On January 1st Mike's mom gives him 1 cent for doing his chores. Each day, after the first day, she pays him double the amount she paid him the day before. On January 13th she pays him for the last time deciding it was getting too expensive. How much money did Mike earn in total during those 13 days? Give your answer in dollars with two decimal places.

Answers
1. _____
2. _____
3. _____

Solutions to Category 1
# Mystery
Meet #4, February 2009

Answers

1. 15

2. W

3. $81.91

1. If a number is 5 more than a multiple of 6, then it is also 1 less than a multiple of 6. The multiples of 6 which are positive and between 0 and 100 are: 6, 12, 18, 24, ... 96 and there are 16 of these. Subtract 1 from each and you get 5, 11, 17, 23, ...95 of which there are still 16 numbers. However 5 is 5 greater than 0 which is not positive and can no longer be counted. Therefore there are 15 integers between 1 and 100 that are 5 more than a positive multiple of 6. Of course, we could have added 5 to each of the 16 numbers instead but 96+5 = 101 and we'd still have to remove 1 from the 16 numbers.

2. Every three letters of the alphabet are written in groups of 6 letters. So the first three letters are the first 6 in the pattern. Since 42 is the closest multiple of 6 to 45 we can find that 42 = 6 x 7. After 7 sets of 3 letters (21 letters of the alphabet) the pattern would have 42 letters in it. The 21st letter of the alphabet is U. So U is written 3 times and the last time is the 42nd letter in the pattern. The 43rd letter would be V, and the 44th and 45th letters in the pattern would be W, which is our answer.

3. After 13 days Mike would have earned:
$1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 + 2048 + 4096 = 8192 - 1 = 8191$ cents = \$81.91. The sum of the powers of 2 starting with 1 is always 1 less than the next largest power of 2. In this case we are adding from 2^0 up to 2^{12} and the sum will be $2^{13} - 1 = 8192 - 1 = 8191$ cents = \$81.91.
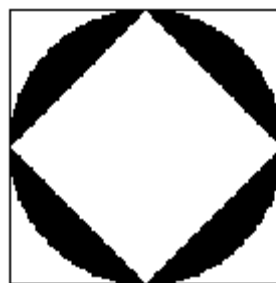
Category 2
Geometry
Meet #4, February 2009

1. How many square centimeters are in the area of a circle which has a diameter of 12 cm? Express your answer to the nearest thousandth of a square centimeter. Use 3.141592 as an estimation for π.

2. In the figure below, two congruent circles pass through each other's center. $\overline{AC}$ is the diameter of the circle with center B and A is the center of the other circle. Point D is one of the intersection points of the two circles. How many degrees are there in $\angle$DBC?
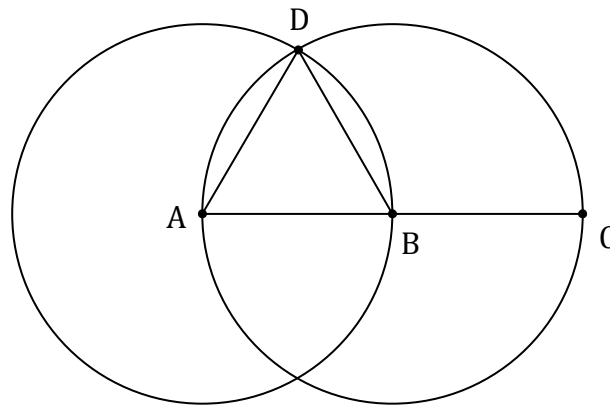
3. In the diagram below, a circle is inscribed in a square with side lengths 14 cm. A smaller square is inscribed in the circle. The area inside the circle but outside the inner square is then shaded. What is the area of the white part of the diagram? Use 3.14159 for π and express your answer as a decimal to the nearest hundredth.

Answers

1. _____

2. _____

3. _____

Solutions to Category 2
# Geometry
Meet #4, February 2009

**Answers**

1. 113.097

2. 120

3. 140.06

1. The formula for the area of a circle is $A = \pi r^2$. If the diameter of the circle is 12 cm, then the radius is 6 cm. The area of the circle is then $A = \pi 6^2 = 36\pi \approx 36(3.141592) \approx 113.0973 \approx 113.097$.

2. By drawing in the segment AD, we have an equilateral triangle ABD since all three sides are radii of two circles which are congruent. Since the triangle is equilateral, ∠ABD is 60°. We also know that ∠DBC is supplementary to ∠ABD, so ∠DBC = $180 - 60 = 120°$
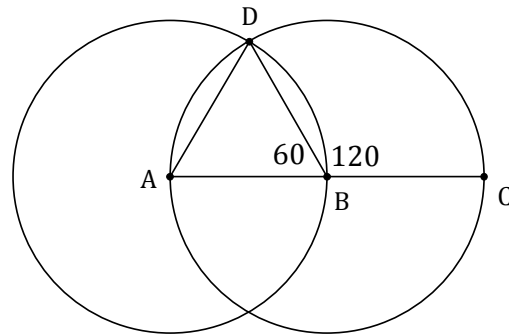
3. The area of the largest square is $14^2 = 196$ cm². The diameter of the circle is 14, so its radius is 7 and its area is $\pi 7^2 = 49\pi \approx 49(3.14159) \approx 153.937 \approx 153.94$ cm². The smaller square is exactly half of the larger square, as shown below, so its area is 98 cm². The area of the white region would then be $196 - 153.94 + 98 = 140.06$ cm².
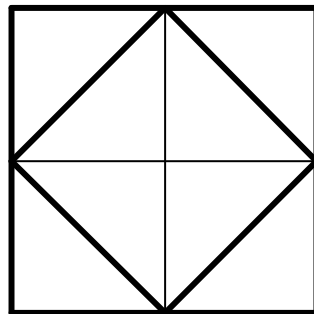
Category 3
Number Theory
Meet #4, February 2009

1. Mike's birthday is on Friday the 13th this year. Bob's next birthday is 180 days after Mike's. What day of the week is Bob's next birthday?

2. The 7th and 13th terms of an arithmetic sequence are -45 and 21, respectively. What is the 50th term in this sequence?

3. What is the greatest value of $M + N$ if M and N are single digit numbers, given the modular congruence below?

$$11M + 7N \equiv 23 \ (\text{mod } 4)$$

Answers
1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #4, February 2009

Answers

1. Wednesday

2. 428

3. 17

1. 180 days is equal to 25 full weeks and 5 more days. 25 full weeks after Mike's birthday would be still be a Friday, and 5 days after that would be a Wednesday.

2. The 13th term of the sequence is 6 terms after the 7th term. Since the sequence is arithmetic that means the same number has been added to the 7th term 6 times to get the 13th term. That means that $-45 + 6d = 21$ where d is the number being added each time. Solving that equation gives us $d = 11$. To get to the 50th term in the sequence, 11 will need to be added to the 13th term 37 more times. So the 50th term is $21 + 37(11) = 21 + 407 = 428$.

3.

$$11M + 7N \equiv 23 \ (\text{mod } 4)$$
$$3M + 3N \equiv 3 \ (\text{mod } 4)$$
$$M + N = 1 \ (\text{mod } 4)$$

So $M + N$ has a remainder of 1 when divided by 4. That means $M + N$ could be 1, 5, 9, 13, 17, 21, 25, ..., but the largest $M + N$ could possibly be if they are both single digit numbers is 18. Therefore the greatest value of $M + N$ that satisfies this modular congruence is 17.

Category 4
Arithmetic
Meet #4, February 2009

1. The Euclidean Middle School math team had a score of 120 at the 1st IMLEM meet of the year and a score of 165 at the 2nd meet of the year. What was the percent increase in the math team's score from the 1st meet to the 2nd meet? Express your answer as a mixed number.

2. Akeelah bought a dictionary that was on sale for 27% off. The sale price was $35.04. What was the original price of the dictionary in dollars?

3. Frank went to the bank to deposit $6,000 in a new savings account. The bank had two offers for him. He could either earn 6% annual interest compounded monthly, or 7% annual interest compounded annually. At the end of 3 years, how much more interest will he have earned if he chooses the better offer than if he had chosen the worse offer? Assume no withdrawals or additional deposits were made during these three years. Round your answer to the nearest whole dollar.

Answers
1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #4, February 2009

Answers

1. $37\frac{1}{2}$

2. $48

3. $170

1. The team's score increased by 45 points which is compared to the original score of 120 to find the percent increase.

$$\frac{45}{120} = \frac{3}{8} = 37\frac{1}{2}\% \text{ increase}$$

2. If Akeelah received 27% off, that means she still had to pay for 73% of the original price. Since she paid $35.04 we can use the following equation to find the original price, x:

$$.73x = 35.40$$
$$x = 48$$

3. If Frank chooses to compound the interest monthly he would use this equation to find the value after 3 years:

$$V = 6000 \times \left(1 + \frac{.06}{12}\right)^{3 \times 12} = 6000(1.005)^{36} \approx 7180.083$$

If Frank decides to compound the interest annually he would use this equation to find the value after 3 years:

$$V = 6000 \times (1 + .07)^3 = 6000(1.07)^3 = 7350.258$$

Using the second option he would have earned $7350.258 - 7180.083 = 170.175 \approx$ 170 more.

Category 5
Algebra
Meet #4, February 2009

1. It took Joey 6 days to assemble the 32 ginger bread houses he planned to sell at the local fair. If he assembles gingerbread houses at the same rate as he did for the local fair, how many days will it take him to assemble the 112 ginger bread houses he plans to sell at the state fair?

2. Billy's teacher has told her students that the value of a word can be found by assigning a fixed number of points for each consonant and a different fixed value for each vowel and finding the sums of the values for the whole word (all consonants are worth the same as each other and all vowels are worth the same as each other, but consonants and vowels are worth a different amount from each other). If Billy finds that the value of the word "math" is 35 points and the value of the word "league" is 60 points, then what is the value of the word "Intermediate"? (A, E, I, O, U are the vowels, all other letters here are consonants.)

3. Jan has 2 animals, Pat the cat and Snog the dog. If both animals gained 20 pounds, Snog would be 25% heavier than Pat. If they both lost 20 pounds, Snog would be 50% heavier than Pat. What is the combined weight, in pounds, of Pat and Snog?

Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #4, February 2009

Answers

1. 21

2. 114

3. 140

1. Since it took Joey 6 days to build 32 houses that gives us the ratio $\frac{6 \text{ days}}{32 \text{ houses}}$. He needs to build 112 houses and we want to know the number of days, call that d. We can write a second ratio and form a proportion now: $\frac{6}{32} = \frac{d}{112} \rightarrow \frac{3}{16} = \frac{d}{112} \rightarrow \frac{3 \times 7}{16 \times 7} = \frac{21}{112} = \frac{d}{112}$ So $d = 21$ days.

2. "Math" has 3 consonants and 1 vowel, while "league" has 2 consonants and 4 vowels. Using c as the value of a consonant, and v as the value of a vowel we can write these two equations:
$3c + 1v = 35$
$2c + 4v = 60$
Multiplying the first equation by 4 results in these two equations:
$12c + 4v = 140$
$2c + 4v = 60$
Subtracting the two equations we get:
$10c = 80$ which tells us that a consonant is worth 8 points. Using the original equation for "math" we have $3(8) + v = 35 \rightarrow v = 11$ so vowels are worth 11 points. "Intermediate" has 6 consonants and 6 vowels for a value of $6(8) + 6(11) = 48 + 66 = 114$ points

3. Using P for Pat's weight and S for Snog's weight we get the equations:
$1.25(P + 20) = S + 20$ and $1.5(P - 20) = S - 20$
$1.25P + 25 - 20 = S$ and $1.5P - 30 + 20 = S$
$1.25P + 5 = S$ and $1.5P - 10 = S$
Substituting for S we get:
$1.25P + 5 = 1.5P - 10$
$15 = .25P$
$60 = P$
Plugging 60 in for P in the first equation we get
$1.25(60 + 20) = S + 20$
$1.25(80) = S + 20$
$100 = S + 20 \rightarrow 80 = S$

So the combined weight is $60 + 80 = 140$ lbs

Category 6
Team Questions
Meet #4, February 2009

1. The price of a bicycle is increased by 30% then put on sale for 40% off. At the time of purchase a 5% tax is added on. If the final price of the bicycle is $204.75, what was the original price before any of the three changes?

2. How many triangles are there in which the three vertices are all chosen from among the array of dots to the right?
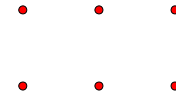
3. How many positive integers less than 1000 can be expressed as x^y where x and y are positive integers and $y > 1$?

4. Barbara works at a donut shop where she sells donuts, muffins and bagels. The first customer of the day buys 3 donuts, 6 muffins and 4 bagels and the total cost is $11. The second customer buys 4 donuts, 3 muffins and 3 bagels and the total cost is $8. The third customer buys 5 donuts, 4 muffins and 6 bagels and the total cost is $13. If the fourth customer buys 2 donuts, 2 muffins and 2 bagels, what will the total cost be for the fourth customer?

5. In the diagram to the right, ABCDE is a pentagon. Point F is the midpoint of side AE. A circle with center G and radius 6 inches is externally tangent to the pentagon at point F. If the circle were to roll around the perimeter of the pentagon, always remaining tangent to the pentagon, how many inches would point G have travelled when the circle was again tangent to the pentagon at point F given that the perimeter of the pentagon is 100 inches? Round your answer to the nearest whole number.
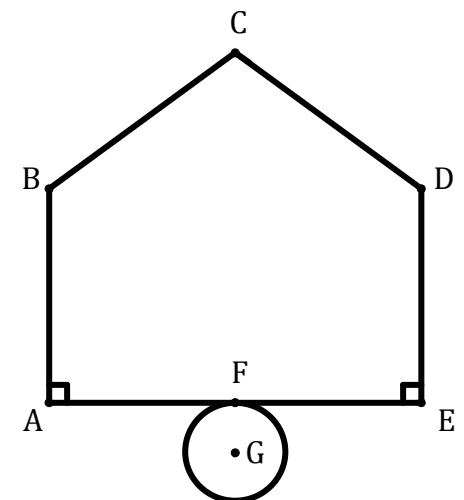
Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____ = E
6. _____

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below.

$$\sqrt{D \cdot \sqrt{\frac{C+E-B}{A}}}$$

Solutions to Category 6
Team Questions
Meet #4, February 2009

Answers

1. 250

2. 18

3. 40

4. 5

5. 138

6. 2

**1.** This equation will find the original price:
$$1.05(.6(1.3x)) = 204.75$$
$$.819x = 204.75$$
$$x = 250$$

2. If you draw them out, there are 18. You could also use combinations to do this. There are 6 points and you need to choose 3 of them as vertices. That's $_6C_3 = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$ ways to pick 3 points, but 2 of those ways results in three points which are horizontal and form a line segment not a triangle. That's $20 - 2 = 18$ again.
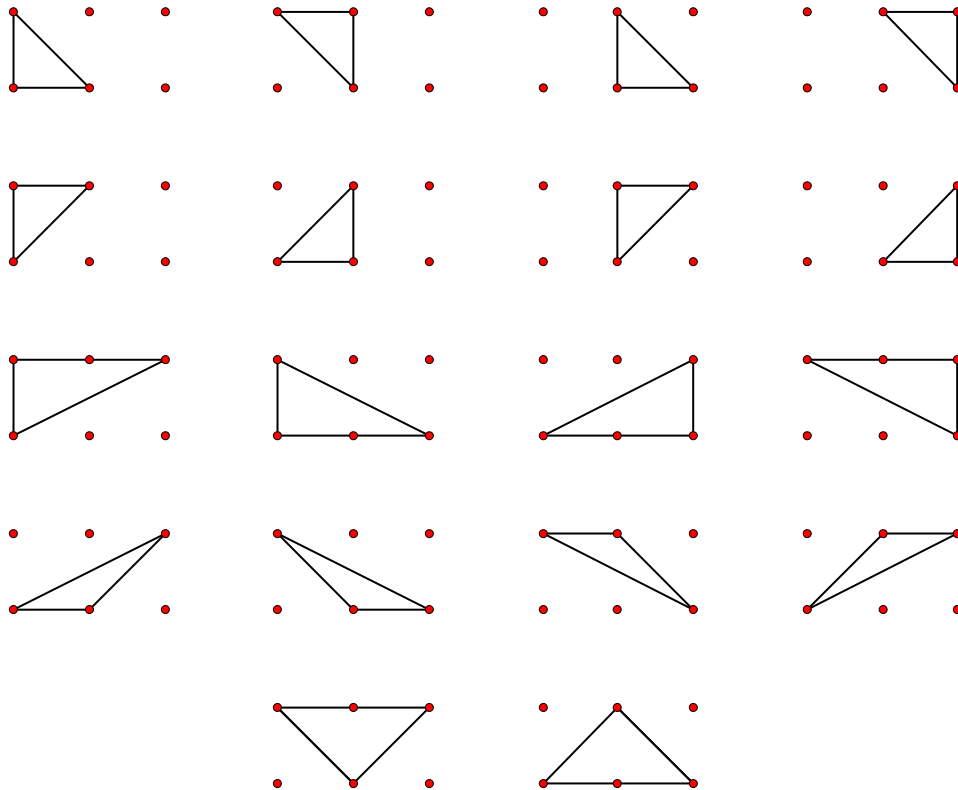
3. 1^2(that's 1), $2^2 \to 2^9$ (8 more), $3^2 \to 3^6$ (5 more), (no powers of 4, they were all counted by powers of 2), $5^2 \to 5^4$ (3 more), $6^2 \to 6^3$ (2 more), $7^2 \to 7^3$ (2 more), (no powers of 8 or 9, they were all counted by powers of 2 and 3), $10^2 \to 31^2$(excluding 16^2, 25^2 and 27^2, which were already counted by 2^8, 5^4 and 3^6, so that's 19 more).

That's a total of $1 + 8 + 5 + 3 + 2 + 2 + 19 = 40$ that can be expressed as x^y.

4. We can write a system of 3 equations with 3 variables, using d for the cost of a donut, m for the cost of a muffin and b for the price of a bagel.

$3d + 6m + 4b = 11 \ \to \ \times 3 \to \ \ 9d + 18m + 12b = 33$
$4d + 3m + 3b = 8 \ \ \ \to \ \times 4 \to 16d + 12m + 12b = 32$
$5d + 4m + 6b = 13 \to \ \times 2 \to \ 10d + \ \ 8m + 12b = 26$

Subtracting the first equation from the second to get one new equation, and subtracting the third equation from the second gives these two new equations.

$7d - 6m = -1 \to \times 2 \to 14d - 12m = -2$
$6d + 4m = 6 \ \ \to \times 3 \to 18d + 12m = 18$

Adding the two equations togther gives us $32d = 16 \to d = 0.5$

Plugging in 0.5 for d into the equation $6d + 4m = 6$ gives us $6(.5) + 4m = 6 \to 3 + 4m = 6 \to 4m = 3 \to m = .75$

Plugging in $.75$ for m and $.5$ for d into the first equation gives us:

$3(.5) + 6(.75) + 4b = 11 \to 1.5 + 4.5 + 4b = 11 \to 6 + 4b = 11 \to 4b = 5 \to b = 1.25$

So two of each would cost $2(.5) + 2(.75) + 2(1.25) = 1 + 1.5 + 2.5 = \$5$

5. The new, lighter lines in the diagram below show the path of the point G. The sum of the new straight lines is equivalent to the perimeter of the pentagon so they add up to 100 inches. The 5 arcs that occur at each vertex of the pentagon would be impossible to find separately without knowing the angles at each vertex, but they will always add up to 360 degrees if the polygon is convex. So the arcs form a complete circle and we need that circle's circumference. The formula for circumference is $C = \pi d = \pi 12 \approx 3.14 \times 12 \approx 37.68 \approx 38$. The total distance the point G travels is $100 + 38 = 138$ inches. (To prove the 5 arcs form a complete circle label all the angles, many of which are right angles, and remember that the sum of the interior angles of a pentagon is $540°$ and each vertex has $360°$ going all the way around it. You'll find that the five missing angles that make the arcs add up to $360°$.)
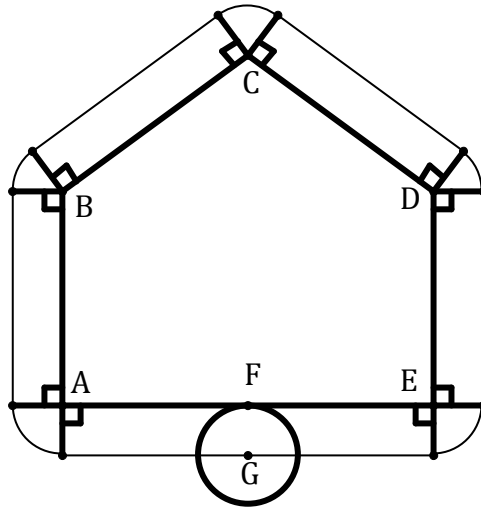
6. $\sqrt{D \cdot \sqrt{\dfrac{C+E-B}{A}}} = \sqrt{5 \cdot \sqrt{\dfrac{40+138-18}{250}}} = \sqrt{5 \cdot \sqrt{\dfrac{160}{250}}} = \sqrt{5 \cdot \sqrt{\dfrac{16}{25}}} =$

$\sqrt{5 \cdot \dfrac{4}{5}} = \sqrt{4} = 2$