Meet #2
December 2008

Intermediate Mathematics League of Eastern Massachusetts

Meet #2
December 2008

Category 1
Mystery
Meet #2, December 2008

1. Mike made 85% of the field goals he kicked for the football team. If he attempted 60 field goals, how many did he miss?

2. The pages of a book are numbered consecutively from 1 to 275. How many times is the digit 8 used in numbering the pages?

3. When their mother turned 40, Sean was four times as old as his brother. When their mother turned 48, Sean was twice as old as his brother. How old will Sean be when their mother turns 60?

Answers

1. _____
2. _____
3. _____

Solutions to Category 1
Mystery
Meet #2, December 2008

Answers

1. 9

2. 47

3. 36

1. If Mike made 85% of the field goals he attempted, then he missed 15%. There are several ways to find 15% of 60. One way is to multiply .15 by 60 which is 9. You could also say that 10% of 60 is 6 and 5% of 60 is 3 (half of 6), so 15% of 60 is 6 + 3 = 9.

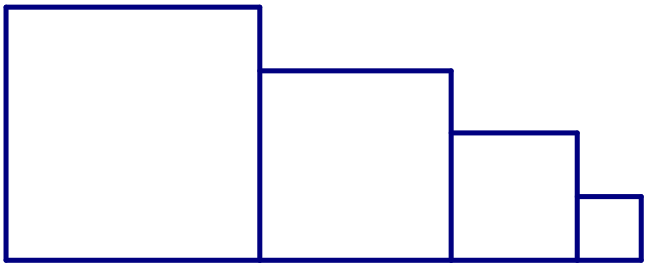
2. From 1 to 99 the digit 8 is used 20 times (10 times in the units place and 10 times in the tens place). From 100 to 199 the digit 8 is again used 20 times. From 200 to 275 the digit 8 is used 7 times (once each in 208, 218, 228, 238, 248, 258, and 268). That's a total of 20 + 20 + 7 = 47 times the digit 8 is used.

Editor note: One might argue that "How many times is the digit 8 used in numbering the pages?" could be answered by going through the pages, asking for each one, "is the digit 8 used in numbering this page?" Yes or no. Count how many times you said Yes. This would give 45 as an answer, which is the number of pages rather than the number of 8's. If this were the intent of the problem, the given wording would be an extremely ambiguous way to word it. Compare to "How many page numbers contain the digit 8?" See Occam's Razor.

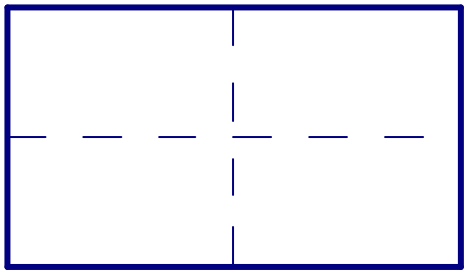
3. When the mom was 40 Sean was four times as old as his brother which gives us S = 4B (where S is Sean's age then and B being Bob's age then). Eight years later he was only twice as old, so S + 8 = 2(B+8). Replacing S with 4B in the second equation you get 4B + 8 = 2B + 16. So B = 4. B represented the brother's age when mom was 40, so Sean would have been 4B = 4(4) = 16 when his mom was 40. Twenty years later, when their mom is 60, Sean will be 36.

Category 2
Geometry
Meet #2, December 2008

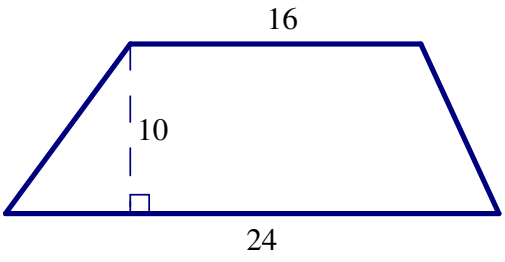
1. The four squares below have areas of 16, 9, 4, and 1 respectively. The squares are lined up one next to the other as shown below. What is the perimeter of the overall shape below?

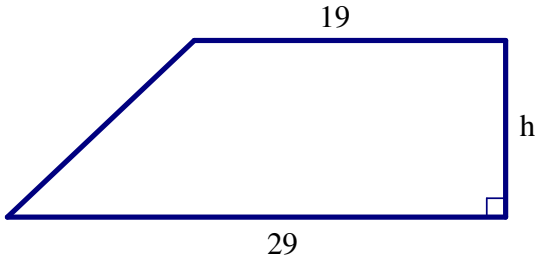


2. A rectangle with a perimeter of 60 inches is cut into 4 smaller congruent rectangles by cutting the rectangle in half both horizontally and vertically as shown to the right. How many inches are in the sum of the perimeters of the four new rectangles?



3. The trapezoid on the right has an area that is 16 more than the area of the trapezoid on the left. What is the value of h (the height) in the trapezoid on the right?





Answers

1. _____
2. _____
3. _____

Solutions to Category 2
Geometry
Meet #2, December 2008

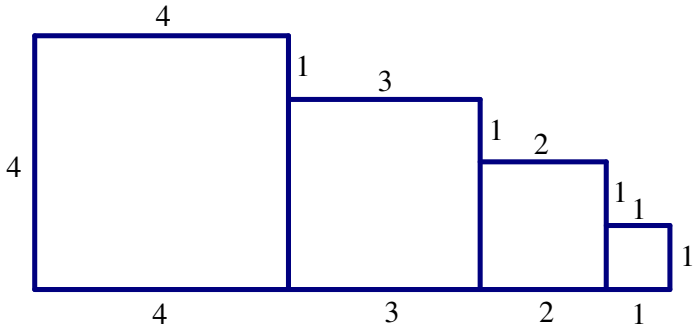
Answers

1. 28

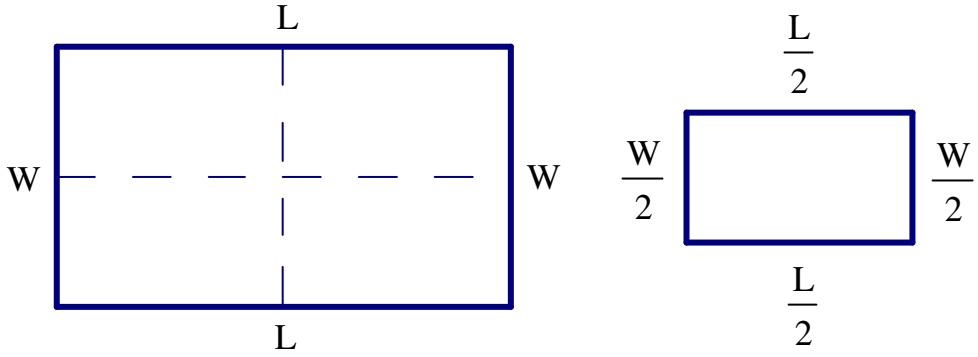
2. 120

3. 9

1. The total perimeter of the figure is:
4 + 1 + 3 + 1 +2 + 1 + 1 + 1 + 1 + 2 + 3 + 4 + 4 = 28



2. Labeling the rectangles' length and width with L and W we can call the perimeter of the original rectangle 2W + 2L. Each of the smaller rectangles would have a length of $\frac{L}{2}$ and a width of $\frac{W}{2}$ giving them each a perimeter of W + L and the four of them combined would be 4W + 4L. That's exactly twice the perimeter of the original, so the combined perimeter of the four rectangles must be 2(60) = 120



3. The area of the trapezoid on the left is $A_{Trapezoid} = \frac{(b_1+b_2)\cdot h}{2} = \frac{(16+24)\cdot 10}{2} = \frac{400}{2} = 200$. So the area of the trapezoid on the right is 200+16 = 216. Using the formula for area of a trapezoid we can find the missing height given the area.

$$216 = \frac{(19+29)\cdot h}{2}$$
$$216 = \frac{48h}{2}$$
$$216 = 24h$$
$$9 = h$$

Category 3
Number Theory
Meet #2, December 2008

1. What is the Greatest Common Factor of 216 and 504?

2. The prime factorization of X is $a^b \times c^d$. None of the values of $a, b, c,$ or d is equal to 2. Only one of the four values is 1. All four values of $a, b, c,$ and d are different. What is the smallest possible value of X?

3. The Least Common Multiple of positive integers a and b is 144. The Greatest Common Factor of a and b is 12. What is the value of $ab \div 27$?

Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #2, December 2008

Answers

1. 72

2. 405

3. 64

1. $216 = 2^3 \times 3^3$ and $504 = 2^3 \times 3^2 \times 7^1$
The Greatest Common Factor of 216 and 504 is equal to $2^3 \times 3^2 = 72$.

2. Since none of the values can be equal to 2, a and c should be 3 and 5 if we want X to be as small as possible (they can't be 1 since 1 isn't prime). Since we can use 1 just once, using it as the exponent of 5 will keep X smallest. That leaves the exponent for 3. The smallest value left to use is 4. So X = $3^4 \times 5^1 = 81 \times 5 = 405$.
Editor note: We can't use zero as no proper "prime factorization" has a zero as either a base or an exponent. Otherwise, there would be no unique prime factorization: 10 = $2^1 \cdot 5^1$ or $2^1 \cdot 5^1 \cdot 7^0$, etc. See the Fundamental Theorem of Arithmetic.

3. The LCM(a, b) times the GCF(a, b) will always be equal to the product of a and b. So $ab = 12 \times 144 = 1728$ and $ab \div 27 = 1728 \div 27 = 64$. A faster way to compute that would be $\frac{12 \times 144}{27} = \frac{4 \times 144}{9} = \frac{4 \times 16}{1} = 64$.

Category 4
Arithmetic
Meet #2, December 2008

1. What is the fraction that is equivalent to the decimal .3125? Express your answer as a common fraction. (A "common fraction" is written in the form $\frac{M}{N}$ where $\frac{M}{N}$ cannot be simplified. A "mixed number" is not a common fraction.)

2. How much larger is $37\frac{1}{2}\%$ of 96 than $\frac{5}{12}$ of 84?

3. What is the 80^{th} digit to the right of the decimal point in the repeating decimal equivalent of $\frac{3}{70}$?

Answers

1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #2, December 2008

Answers

1. $\frac{5}{16}$

2. 1

3. 4

1. $.3125 = \frac{3125}{10000} = \frac{625}{2000} = \frac{125}{400} = \frac{25}{80} = \frac{5}{16}$

2. $37\frac{1}{2}\%$ of $96 = \frac{3}{8} \times 96 = 36$

$\frac{5}{12}$ of $84 = \frac{5}{12} \times 84 = 35$

36 is 1 larger than 35

3. $\frac{3}{70} = 0.0\overline{428571}$. The repeating part of the decimal has 6 digits, but due to the 0 at the beginning, each set of 6 repeating digits would end at the 7^{th}, 13^{th}, 19^{th}, ……67^{th}, 73^{rd}, and 79^{th} place after the decimal. So the 79^{th} place after the decimal is 1 and the 80^{th} place after the decimal is 4.

Category 5
Algebra
Meet #2, December 2008

1. The sum of 5 consecutive odd numbers is 105. What is the largest of the 5 numbers?

2. Shandra and Terri are sisters who were both given the same amount of money by their mother. Shandra was able to triple her money doing chores, while Terri spent 6 of her dollars. Shandra now has 4 times as much money as Terri. How many dollars did Terri's mother give her?

3. The sum of the first n natural numbers is known as the n^{th} triangular number. The formula for the n^{th} triangular number is $T_n = \frac{n(n+1)}{2}$. The sum of the first n cubic numbers is equal to the n^{th} triangular number squared. If the sum of the first n cubes is 6084, what is the value of n?

Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #2, December 2008

Answers

1. 25

2. 24

3. 12

1. By calling the middle of the five odd numbers x, we can use the equation below to find the middle number

$(x-4)+(x-2)+x+(x+2)+(x+4)=105$

$5x=105 \rightarrow x=21$

If the middle number is 21, then the five numbers are 17, 19, 21, 23, 25.

2. Calling the amount of money Terri and Shandra started with S, we can use the equation $3S=4(S-6) \rightarrow 3S=4S-24 \rightarrow S=24$.

3. The sum of the first *n* cubes is equal to $(T_n)^2$, so the first thing we need to figure out is the square root of 6084. We can estimate this by noticing that $80^2=6400$ so we know the square root is less than 80. Since the number ends in 4 its square root must end in 2 or 8. So the square root of 6084 is either 72 or 78 (since we know it is an integer). $72^2 = 5184$ and $78^2 = 6084$. We now know that $T_n = 78 = \frac{n(n+1)}{2} \rightarrow 156 = n(n+1) \rightarrow 12 \cdot 13 = n(n+1) \rightarrow 12 = n$

Category 6
Team Questions
Meet #2, December 2008

1. Mike counts by threes starting with 200 (200, 203, 206, … etc). Ramille counts backwards by twos starting with 800 (800, 798, 796, … etc). If they each start at the same time and wait exactly 5 seconds between saying their next number, what number will they say at the same time?

2. A regular pentagon, a regular hexagon, and a regular octagon all have different positive integer side lengths and the same perimeter. What is the smallest possible perimeter that they could all have?

3. What is the smallest positive integer with exactly 9 positive factors?

4. If 3 chickens can lay 2 dozen eggs in 4 days, how many days will it take 7 chickens to lay 3 and a half dozen eggs?

5. How many pairs of positive integers, where the two integers are not necessarily distinct, have a Least Common Multiple of 24? *Note: the pairs (1, 24) and (24, 1) should be considered the same and only counted once.

Answers

1. _____ = *A*
2. _____ = *B*
3. _____ = *C*
4. _____ = *D*
5. _____ = *E*
6. _____

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below.

$$\frac{C-\frac{AD}{B}}{E}$$

Solutions to Category 6
Team Questions
Meet #2, December 2008

Answers

1. 560

2. 120

3. 36

4. 3

5. 11

6. 2

1. $200 + 3x = 800 - 2x$
$5x = 600$
$x = 120$
$200 + 3(120) = 200 + 360 = 560$
$800 - 2(120) = 800 - 240 = 560$

2. The perimeter of a regular pentagon with integer side lengths would be $5x$, the perimeter of a regular hexagon would be $6y$, and the perimeter of an octagon would be $8z$ (where x is the side length of the pentagon, y is the side length of the hexagon, and z is the side length of the octagon). The perimeters of the polygons would then be multiples of 5, 6, and 8 and the Least Common Multiple of 5, 6, and 8 is 120.

3. For a positive integer to have an odd number of positive integer factors, it must be a perfect square. So we could try looking at just the square numbers until we find that 36 has 9 factors (1, 2, 3, 4, 6, 9, 12, 18, 36). We could also use the formula for how many factors a number has and work backwards. Any number with a prime factorization of $x^a \cdot y^b$ will have a total of $(a + 1)(b + 1)$ factors. For a number to have 9 factors $(a + 1)(b + 1)$ must equal (3)(3) or (1)(9). So either a and b are both 2 or one of them is 0 and the other is 8. If a and b are both 2, the smallest number would have a prime factorization of $2^2 \cdot 3^2 = 36$. If a and b are 0 and 8 then the smallest number would have a prime factorization of $2^8 \cdot 3^0 = 256$. So 36 is the smallest positive integer with 36 positive integral factors.

4. If 3 chickens need 4 days to lay 24 eggs, then 1 chicken would need 12 days to lay 24 eggs, so 1 chicken can lay 2 eggs per day. Seven chickens then would lay 14 eggs in one day. Since we want to know how long it will take for them to lay 42 eggs they would need $42 \div 14 = 3$ days.

5. The following pairs, listing the larger number first, (to avoid double counting pairs) are the possible pairs of numbers with a LCM of 24.

(24, 1)
(24, 2)
(24, 3)
(24, 4)
(24, 6)
(24, 8)
(24, 12)
(24, 24)
(12, 8)
(8, 3)
(8, 6)

That is 11 pairs of numbers with a LCM of 24

6.

$$\frac{C-\frac{AD}{B}}{E} = \frac{36-\frac{560\cdot 3}{120}}{11} = \frac{36-\frac{1680}{120}}{11} = \frac{36-14}{11} = \frac{22}{11} = 2$$