Intermediate
Mathematics League
of
Eastern Massachusetts

# Category 1
Mystery
Meet #4, February 2008

1. At Bob's Cycle Emporium he sells Cycles that all have one seat, but have various numbers of wheels. Currently he only has tricycles (which have three wheels) and pentacycles (which have 5 wheels) in stock. Bob counts, and finds that there are a total of 42 seats and 176 wheels in the store. How many pentacycles does Bob have in stock at the store?

2. The first five stages of a sequence are shown below. If the pattern continues, how many shaded squares will there be in the 17th stage?

1st Stage 2nd Stage 3rd Stage 4th Stage 5th Stage
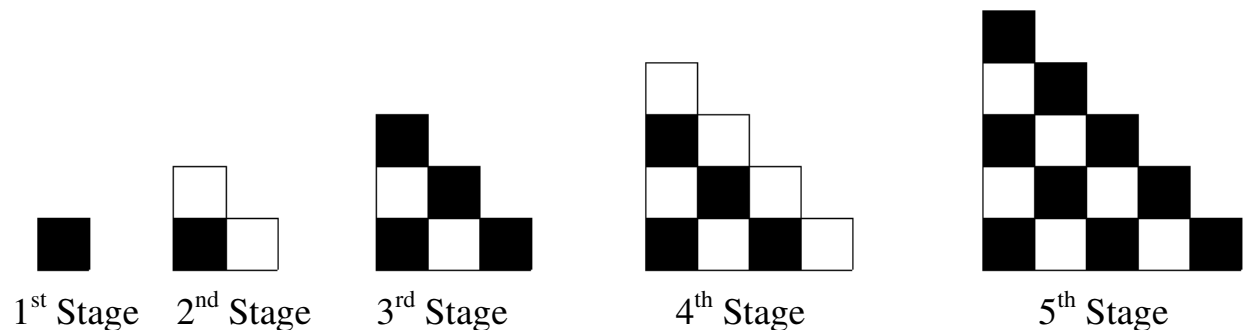
3. Sarah and Dominic both like to count. Sarah starts with 4 and counts by fours. Dominic starts with 6 and counts by sixes. If they both stop just after saying 192, what fraction of the numbers that Sarah said did Dominic also say? Express your answer as a simplified fraction.

Answers

1. _____

2. _____

3. _____

Solutions to Category 1
Mystery
Meet #4, February 2008

1. **W**e can write two equations using t for the number of tricycles and p for the number of pentacycles.

$$t + p = 42$$
$$3t + 5p = 176$$

Multiplying the first equation by 3 and subtracting it from the second we get $2p = 50$ so $p = \mathbf{25}$

2. The first two stages each have 1 shaded square, the next two stages have 4 shaded squares, and the next two have 9. You can continue listing those out, or notice that each even numbered stage (and the stage before it) has as many squares as half the stage number squared. So the 17th stage will be the same as the 18th stage which has $\left(\frac{18}{2}\right)^2 = 9^2 = \mathbf{81}$ shaded squares.

In other words, the nth even and n^{th} odd numbered stages all have n^2 shaded squares. Since 17 is the 9th odd number there will be $9^2 = 81$ shaded squares

3. Sarah counts the multiples of four. 192 is 4×48, so Sarah says 48 numbers. The LCM of 4 and 6 is 12, meaning that the numbers that Dominic and Sarah will both say are the multiples of 12. Since $192 = 12 \times 16$, sixteen numbers will be said by both kids. So Dominic says $\frac{16}{48} = \frac{\mathbf{1}}{\mathbf{3}}$ of Sarah's numbers.

Category 2
Geometry
Meet #4, February 2008

You may use a calculator today!

1.　In the semi-circle to the right, Point O is the
center, and Points A and J are at opposite ends of
a diameter. The 8 points B, C, D, E, F, G, H, and
I are equally spaced around the semicircle and each is connected to both ends of
the diameter AJ. What is the sum of the degrees in the measures of the angles
∡ABJ, ∡ACJ, ∡ADJ, ∡AEJ, ∡AFJ, ∡AGJ, ∡AHJ, and ∡AIJ?
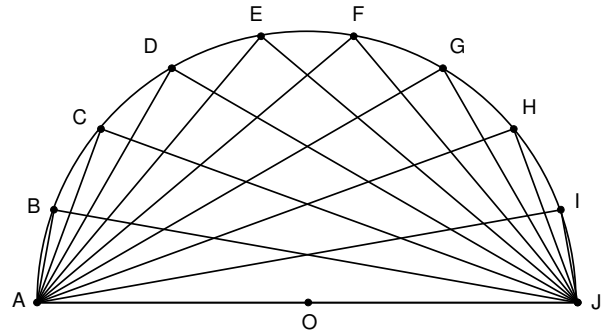
2.　In the square AIDF at the right two semicircles are
drawn using AI and DF as diameters and the two
semicircles are tangent to each other. AF = 8 cm. An ant
crawls along the lines and arcs in a path that takes it from
Point A and then through B, C, D, E, F, G, H, I, J, and back
to A, in that order. How many centimeters long was the
path the ant travelled? Express your answer as a decimal to
the nearest tenth of a centimeter.
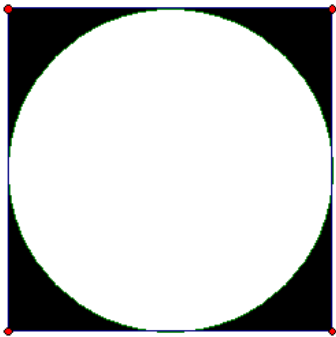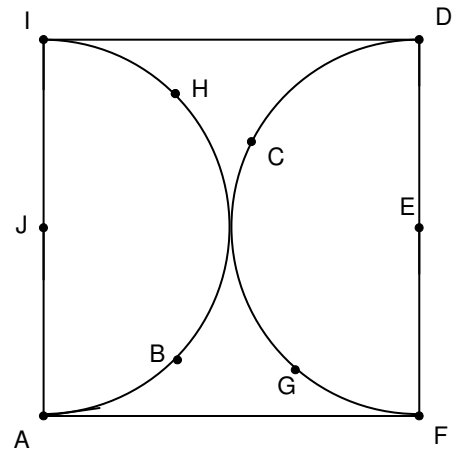
3. A circle is inscribed in the square to the left and the area
between the two shapes is shaded. Using 3.14 as an
estimation for π Jimmy calculated that the area of the
shaded region is 104.06 cm^2. Using 3.14 as an estimation
for π again, how many centimeters are in the circumference
of the circle? Express your answer as a decimal to the
hundredths place.

Answers

1. _____

2. _____

3. _____

Solutions to Category 2
Geometry
Meet #4, February 2008

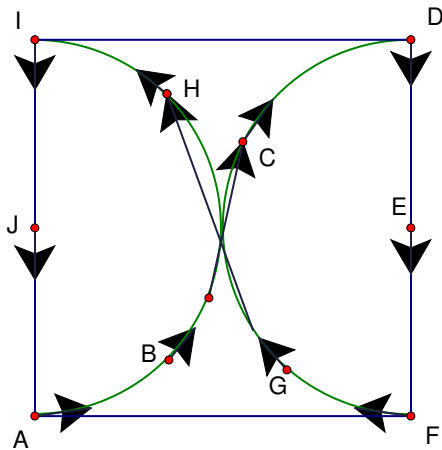**1.** All 8 of the triangles use the diameter as one side and have the 3rd vertex on the circle, so all 8 of the triangles are right triangles. The sum of the angles is just $8 \times 90 = \mathbf{720}$

2. The path the ant travels will be along the entire circumference of the circle once and along the diameter twice. So it will travel a total of $8\pi + 8 + 8 \approx \mathbf{41.1}$

Here is what the path looks like with arrows to guide you.

3. If we call the radius of the circle r, then the sides of the square are all $2r$ making the area of the square $(2r)^2 = 4r^2$. Since the area of the circle is $3.14r^2$, the area of the shaded region must be $4r^2 - 3.14r^2 = .86r^2$ which we know to be 104.06……So……..

$$.86r^2 = 104.06$$
$$r^2 = 121$$
$$r = 11 \text{ and } d = 22$$

Therefore the circumference of the circle is $3.14(22) = \mathbf{69.08}$

Category 3
Number Theory
Meet #4, February 2008

You may use a calculator today!

1. What is the 74th term of this arithmetic sequence?

$$-25, -21, -17, -13, -9, -5, \quad . \, . \, .$$

2. On January 1st, Mike did 8 math problems. Starting January 2nd he began doing 3 more than he did the day before so that he did 11 problems on January 2nd and 14 problems on January 3rd. He continued this each day in January all the way until (and including) January 31st. How many total problems had Mike done during the month of January?

3. For how many positive integer values of x, where $x > 1$, is the following modular congruence true?

$$25 \equiv 1 \ (mod \ x)$$

(Note : That statement is the same as saying 25 equals 1 in Modulo x.)

	Answers
1.	_____
2.	_____
3.	_____

Solutions to Category 3
Number Theory
Meet #4, February 2008

1. The 74th term of the arithmetic sequence would be :

$$-25 + 73(4) = -25 + 292 = \mathbf{267}$$

2. On the first day Mike does 8 problems. For the next 30 days he increase his amount 3 per day so that on January 31st he does 8 + 30(3) = 98 problems.

In total he does 8 + 11 + 14 + 17 + ……. 95 + 98 =
$$\frac{(8+98)31}{2} = \frac{(106)31}{2} = \frac{3286}{2} = \mathbf{1643} \text{ problems}$$

3. In order for $25 \equiv 1 \ (mod \ x)$, when 25 is divided by x the remainder must be 1. That means that x needs to go evenly into 24 so that there will be a remainder of 1 when dividing into 25. So x has to be a factor of 24. There are 8 factors of 24 (1, 2, 3, 4, 6, 8, 12, 24) but x cannot be 1 since x must be greater than 1. That means there are 7 values of x that leave a remainder of 1 when 25 is divided by them, which is the same as saying there are **7** values of x such that $25 \equiv 1 \ (mod \ x)$.

Note : While 25 divided by 1 has a remainder of 0, in Modular arithmetic we can also say that $25 \equiv 0 \equiv 1 \equiv 2 \equiv 3 \equiv \cdots (mod \ 1)$. This is why the exception $x > 1$ was added to the problem.

You may use a calculator today!

1.   Barbara used to live in Maine where she had to pay an 8% tax on all her clothing purchases. She recently retired to Massachusetts where there is no tax at all on clothing. The store she shops at in Massachusetts gives her a 5% senior citizen's discount that the store in Maine did not give her. If Barbara buys a sweater with a price tag of $60 at the Massachusetts store, how much less would it have cost than if she had bought the same sweater in Maine (assuming the $60 price tag was the same)? Express your answer in dollars to the nearest cent.

2.   On Monday, Joe found the T.V. that he wanted to buy priced at $1400. Later that week on Friday he went to purchase the T.V. and the price had dropped to $1008. What percent had the price decreased since Monday?

3.   When Mary was born her Nana put $800 in a bank account for her that earned 5% annual interest to be compounded annually. Her Nana left the account alone until Mary turned 21. How much interest will the account have earned when Mary takes out all the money on her 21st birthday? Express your answer to the nearest whole dollar.

Answers
1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #4, February 2008

Answers

1. 7.80

2. 28

3. 1429

1. In Maine she pays 60(1.08) = 64.80.
In Massachusetts she pays 60(.95) = 57.

That means in Mass. she pays 64.80 – 57 = **7.80** less.

2. If the price fell from $1400 to $1008 that means it decreased
by $392. $\frac{392}{1400}$ = .28, so $392 is **28%** of $1400.

3. For 21 years 5% interest is added at the end of each year. To find the total in
the account after 1 year we can multiply 800 by 1.05 = 840. To find the total at the
end of the second year we would multiply 840 by 1.05. Essentially we will just
multiply 800 by 1.05 a total of 21 times. With a calculator it will be easier to
perform this calculation : $800 \times 1.05^{21} \approx 2229$. Since 800 of that was the
original deposit, 2229 – 800 = **1429** is the interest earned.

You may use a calculator today!

1. Six years ago Monique was three–fourths as old as she is now. How old was she 11 years ago?

2. Captain Bob started crossing a 15 mile pond in his small motor boat which travels at a speed of 16 miles per hour. At some point during the trip the boat's motor died and he had to row the rest of the way at 4 mph. If he spent exactly half the time rowing, how long did the entire trip take? Give your answer in hours and minutes using a colon. (ex. 4:15 for 4 hours 15 minutes)

3. Mike, Ike, and Spike are three friends who put their money together to buy three Red Sox Tickets. Together Mike and Ike have $94. Ike has twice as much as Spike. Mike has $13 more than Spike. If the tickets cost $50 each, how many more dollars do they need?

Answers
1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #4, February 2008

Answers

1. If M is Monique's current age, then : $M - 6 = \frac{3}{4}M$

1. 13

Solving for M we get $M = 24$. So 11 years ago, Monique was $24 - 11 = \mathbf{13}$.

2. 1:30

3. 29

2. Since Captain Bob traveled for the same amount of time both by motor and by rowing, let's call that time t. The distance travelled by motor then is $16t$ and the distance travelled by rowing is $4t$. The total distance travelled then is $16t + 4t = 20t$. We know the total distance travelled is 15 miles, so $20t = 15$ and $t = \frac{3}{4}$ for each mode of travel which is a total of :
$2\left(\frac{3}{4}\right) = 1\frac{1}{2} \text{ hours} = \mathbf{1:30}$.

3. From the information given we can write the following three equations :

$$M + I = 94$$
$$I = 2S$$
$$M = S + 13$$

By substituting the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ equations into the first we get the equation :

$(S + 13) + (2S) = 94 \quad \rightarrow \quad 3S + 13 = 94 \quad \rightarrow \quad 3S = 81 \quad \rightarrow \quad S = 27$

If $S = \$27$, then $I = 2(27) = \$54$ and $M = 27 + 13 = \$40$

They have a total of $\$27 + \$54 + \$40 = \$121$ which is $3(50) - 121 = \mathbf{\$29}$ less than what they need.

Category 6
Team Questions
Meet #4, February 2008

1. Bill, Will, and Phil all paid $204 for the same DVD player at different stores. Bill's cost was after receiving 20% off; Will's cost was after receiving 25% off; and Phil's cost was after receiving 15% off. If instead of receiving the given discounts they all had to pay full price at the different stores, what would the total cost have been for all three DVD players?

2. If the area of a circle is A square inches, the circumference of the same circle is C inches, the radius is a whole number, and 3.14 is used as an estimation of π, then $A - C = 527.52$. What is the diameter of this circle?

3. What is the sum of all positive values of x less than 200 such that :
$$x \equiv 2 \ (mod \ 3) \ ?$$
(In other words, what is the sum of all positive values of x less than 200 such that $x = 2$ in Modulo 3?)

4. Eleven people are standing in a circle and they are all wearing numbers on signs around their necks. The signs are numbered 0 through 10, in order, clockwise around the circle. The person wearing the sign with 1 on it starts counting by saying "one". Then the other members in the circle continue counting clockwise around the circle "two", "three", "four"….except they skip any number divisible by 7 since they think it is unlucky. If they continue in this fashion, what is the number on the sign of the person who says "eighty-five"?

5. How many of the integers from 1 through 51 (inclusive) are not prime, are not the square of an integer, and are not an element of this arithmetic sequence 1, 5, 9, 13 . . . ?

Answers	
1. _____	= A
2. _____	= B
3. _____	= C
4. _____	= D
5. _____	= E
6. _____	

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below.

$$\sqrt{\frac{C-A}{E-1} - 111} + \frac{B}{D}$$

Solutions to Category 6
Team Questions
Meet #4, February 2008

Answers

1. 767

2. 28

3. 6567

4. 7

5. 26

6. 15

1. Bill's $204 was after 20% off, so he paid 80% of the original price which must have been $204 \div .8 = \$255$.

Will's $204 was after 25% off, so he paid 75% of the original price which must have been $204 \div .75 = \$272$.

Phil's $204 was after 15% off, so he paid 85% of the original price which must have been $204 \div .85 = \$240$.

So they _would have_ paid $255 + $272 + $240 = **$767**

2. The area of the circle is $A = 3.14r^2$ and the circumference is $C = 3.14(2r)$.

So $A - C = 3.14r^2 - 3.14(2r) = 3.14(r^2 - 2r) = 527.52$

Dividing by 3.14 on both sides we get :

$r^2 - 2r = r(r - 2) = 168$. We need to find 2 numbers 2 apart that have a product of 168. Using a calculator it is simple to find (by guess and check) that the two numbers are 14 and 12, so $r = 14$. That means the diameter is **28.**

3. The smallest number that has a remainder of 2 when divided by 3 is 2 itself. The next smallest would be 3 bigger, so 5. In fact the numbers that meet this criteria are the members of the arithmetic sequence :

2, 5, 8, 11, ………….., 191, 194, 197

There are 66 numbers in this sequence and the sum of these numbers is :

$$\frac{(2+197)66}{2} = \frac{199(66)}{2} = 199(33) = \textbf{6567}$$

4.   Since they will not say any of the numbers divisible by 7, they will skip the 12 multiples of 7 between 1 and 85 (1,7,14,……84). That means they are only saying 85 – 12 = 73 numbers, so we can treat this problem like they are just counting to 73. If they are counting to 73 without skipping numbers then every multiple of 11 will be said by the person wearing zero, and each number will be said by the person wearing the remainder of that number when divided by 11. So the 73rd number (which is actually 85) will be said by the person wearing #**7** since 73÷11 has a remainder of 7.

Of course you could draw a circle numbered and write the numbers 0 through 10 around it and just count to 85 as you going around the circle while skipping the multiples of 7.

5. First listing the members of the sequence we get:

In the sequence : 1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49 (13 numbers)
The primes : 2, 3, 5̶, 7, 11, 1̶3̶, 1̶7̶, 19, 23, 2̶9̶, 31, 3̶7̶, 41, 43   (9 new numbers)
The squares : 1̶, 4, 9̶, 16, 2̶5̶, 36, 4̶9̶   (3 new numbers)

That's a total of 13+9+3 = 25 numbers that don't count and 51 – 25 = **26** numbers that do.

6.

$$\sqrt{\frac{6567-767}{26-1}} - 111 + \frac{28}{7} =$$

$$\sqrt{\frac{5800}{25}} - 111 + 4 =$$

$$\sqrt{232} - 111 + 4 =$$

$$\sqrt{121} + 4 =$$

$$11 + 4 = \mathbf{15}$$