Meet #3
January 2008

Intermediate Mathematics League of Eastern Massachusetts

Meet #3
January 2008

Category 1
Mystery
Meet #3, January 2008

1. Mike was reading a book when the phone rang. He didn't have a bookmark, so he just remembered that the product of the two page numbers the book was open to was 7482. What is the sum of those two page numbers?

2. Yesterday I said to myself, "Two weeks ago, it was only 3 days until Friday the 13th." What day of the week and day number is it today? (your answer should be of the same form as "Friday the 13th")

3. The beginning of Pascal's Triangle is formed as shown below. The 1 at the top of the triangle is Row 0, the "1 1" below that is Row 1, and the "1 2 1" is Row 2. Each new Row is started and ended with a 1 and the rest of the numbers in the Row are the sum of the two closest numbers from the row above it. If Pascal's triangle is extended to Row 9, how many of the numbers in the triangle will be divisible by 3?

Row 0 → 1
Row 1 → 1 1
Row 2 → 1 2 1
Row 3 → 1 3 3 1
Row 4 → 1 4 6 4 1
•
•
•

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1
Mystery
Meet #3, January 2008

Answers

1. 173

2. Wednesday the 25^{th}

3. 17

1. If the product of the two consecutive integers is 7482, the two numbers must be close to the square root of 7482. By estimating the square root we can quickly find that the two numbers are 86 and 87, and the sum is 86+87 = **173**.

2. Work backwards on this one : If it was three days until Friday the 13^{th}, then it was Tuesday the 10^{th}. If that was two weeks ago, then it was Tuesday the 24^{th}. If that was yesterday then it <u>is</u> **Wednesday the 25th** .

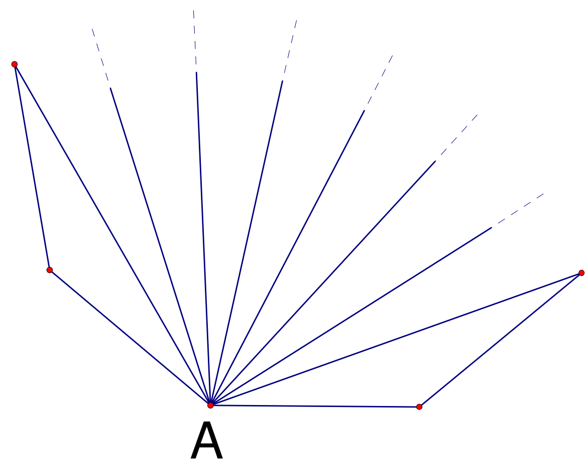
3. Below Pascal's Triangle is extended to the Row 9 and all multiples of 3 are highlighted. There are **17** in total. If you extend the triangle you will find some neat patterns in the appearance of multiples of 3(as well as multiples of other numbers)

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1

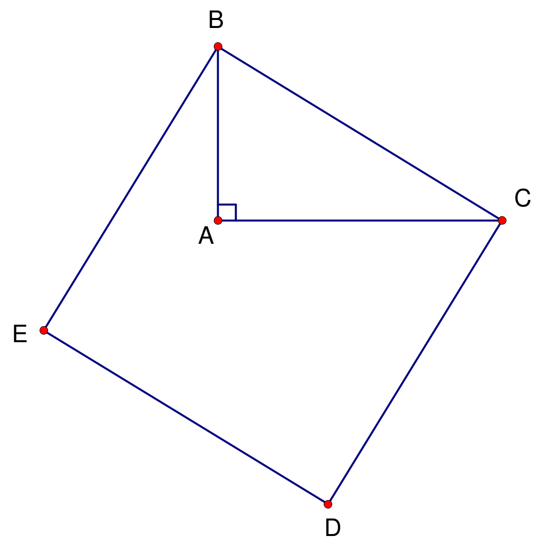
Category 2
Geometry
Meet #3, January 2008

1. If the exterior angle of a regular polygon is 24 degrees, how many sides does the polygon have?

2. In the figure to the right, four of the sides of a regular polygon are shown. Also drawn are all of the diagonals that use A as one of the endpoints of the diagonal; however the full lengths of the diagonals are not always shown. How many diagonals does this polygon have?



3. In the figure below, triangle ABC is a right triangle with right angle at A and quadrilateral BCDE is a square. If AB = $\sqrt{8}$ and AC = $\sqrt{18}$, what is the area of the pentagon ABEDC?



Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 2
Geometry
Meet #3, January 2008

Answers

1. 15

2. 44

3. 20

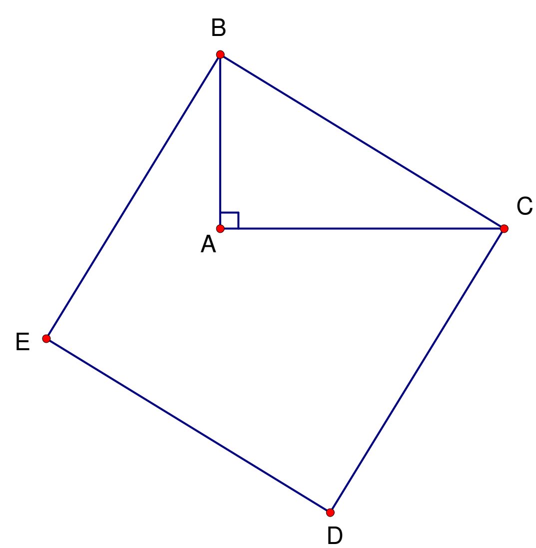
1. Since the sum of the exterior angles is always 360 degrees, if you divide 360 degrees by 24 degrees you get 15 which is the number of equal exterior angles and therefore **15** vertices and sides to the polygon.

2. If there are 8 diagonals and only 2 of the diagonals' other endpoints are shown there must be 6 vertices that are not shown. Since 5 vertices are shown there are a total of 11 vertices and 11 sides. If there are 11 sides there are $\frac{11(11-3)}{2} =$ **44** diagonals total.

3. Using the Pythagorean Theorem:

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 \;\rightarrow\; \overline{BC}^2 = \sqrt{8}^2 + \sqrt{18}^2 \;\rightarrow\; \overline{BC}^2 = 8 + 18 = 26$$

Since $\overline{BC}^2$ is the area of the square, the area of square EBCD is 26. The pentagon we are looking for though is equal to the area of the square minus the area of the right triangle ABC. The area of triangle ABC = $\frac{\sqrt{8}\sqrt{18}}{2} = \frac{\sqrt{144}}{2} = \frac{12}{2} = 6$. The area of pentagon ABEDC = 26 – 6 = **20**.



Category 3
Number Theory
Meet #3, January 2008

1. Express the base 4 number 3213_4 as a base ten number.

$3213_{\text{base four}} = \underline{\qquad\qquad}_{\text{base ten}}$

2. A jar of sand has 910g of sand inside of it. Each grain of sand weighs .013g. How many grains of sand are in the jar? Express your answer in scientific notation.

3. What is the base seven value of this subtraction problem?

$23541_7 - 5635_7 = \underline{\qquad\qquad}_{\text{base 7}}$

Answers
1. $\underline{\qquad\qquad}_{\text{base 10}}$
2. $\underline{\qquad\qquad\qquad}$
3. $\underline{\qquad\qquad}_{\text{base 7}}$

Solutions to Category 3
Number Theory
Meet #3, January 2008

Answers

1. 231

2. 7×10^4
or 7.0×10^4

3. 14603_7

1. $3213_4 = 3(4^3)+2(4^2)+1(4^1)+3(4^0) = 3(64)+2(16)+1(4)+3(1) = 192 + 32 + 4 + 3 =$ **231** base 10

2. $\frac{910}{.013} = \frac{910000}{13} = 70000 = \mathbf{7 \times 10^4 \text{ or } 7.0 \times 10^4}$

3. To subtract 5635_7 from 23541_7 you do the exact same thing you would do if they were both base ten numbers except that when you need to borrow, you do not borrow 10, you borrow a 7. So in the first step of the problem when you need to subtract the 5 from the 1 you would normally borrow from the 4 and add 10 to the 1. Now you will borrow from the 4 and add 7 to the 1. The units digit becomes 8 – 5 = 3. Continuing this subtraction you get $\mathbf{14603_7}$.

Category 4
Arithmetic
Meet #3, January 2008

1. Evaluate the expression below :

$$\sqrt{\sqrt{36}+\sqrt[3]{512}+\sqrt[4]{625}+\sqrt[5]{243}+\sqrt[6]{64}+\sqrt[7]{1}+\sqrt[8]{0}}$$

2. How many multiples of two are there between $\left(1\frac{1}{2}\right)^3$ and $\left(\frac{2}{7}\right)^{-3}$?

3. If $a=\sqrt{6}$ and $b=\sqrt{24}$, what is the value of the expression $(ab)^3$?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 4
Arithmetic
Meet #3, January 2008

Answers

1. 5

2. 20

3. 1728

1. $\sqrt{\sqrt{36} + \sqrt[3]{512} + \sqrt[4]{625} + \sqrt[5]{243} + \sqrt[6]{64} + \sqrt[7]{1} + \sqrt[8]{0}} =$
$\sqrt{6 + 8 + 5 + 3 + 2 + 1 + 0} = \sqrt{25} = \mathbf{5}$

2. $\left(1\frac{1}{2}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8} = 3\frac{3}{8}$
$\left(\frac{2}{7}\right)^{-3} = \left(\frac{7}{2}\right)^3 = \frac{343}{8} = 42\frac{7}{8}$
Between those two mixed numbers are the even numbers from 4 to 42.
From 4 to 42 there are **20** multiples of 2:

4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42

3. $\left(\sqrt{6} \times \sqrt{24}\right)^3 = \left(\sqrt{144}\right)^3 = 12^3 = \mathbf{1728}$

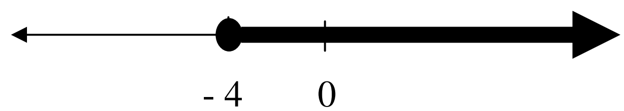
Category 5
Algebra
Meet #3, January 2008

1. What is the positive difference between the two solutions to this equation?

$$|6-3x|=12$$

2. For what value of *K* is the solution set of the inequality below given by the graph below?

$$-2(3x-7) \leq K - 2x + 6$$



3. For how many integer values of *y* is $\left|\dfrac{18}{2-y}\right|$ a positive integer?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5
Algebra
Meet #3, January 2008

Answers

1. 8

2. 24

3. 12

1. $|\mathbf{6-3x}| = \mathbf{12}$

$6-3x = 12 \text{ or } -12$

$-3x = 6 \text{ or } -18$

$x = -2 \text{ or } 6$

The positive difference between – 2 and 6 is 6 – (– 2) = **8**

2. $-2(3x-7) \le k - 2x + 6$

$-6x + 14 \le k - 2x + 6$

$-4x + 8 \le k$

This inequality becomes an equality at the border point $x = -4$

$-4(-4) + 8 = k = 16 + 8 = \mathbf{24}$

3. Since y is an integer, $2-y$ will be an integer. In order for $\left|\frac{18}{2-y}\right|$ to be a positive integer, 18 must be divisible by $2-y$ but you don't need to worry about the positive part since the absolute value bars will take care of that. Basically we want to know how many values of y make $2-y$ a factor of 18 remembering that $2-y \ne 0\ so\ y \ne 2$. So $2-y = 18, 9, 6, 3, 2, 1, -1, -2, -3, -6, -9, -18$ which means that $y = -16, -7, -4, -1, 0, 1, 3, 4, 5, 8, 11, 20$ for **12** integer values.

Category 6
Team Questions
Meet #3, January 2008

1. A "not quite" Pythagorean Triangle is a triangle with integer side lengths in which the sum of the squares of the two shorter sides is one more than the square of the longest side. What is the least possible perimeter of a "not quite" Pythagorean Triangle if the triangle is also scalene? (note : in a scalene triangle all sides are different lengths)

2. Convert the base ten number 84535 into a base 7 value.

3. What is the smallest number of sides a regular polygon could have so that the number of diagonals in the polygon is greater than the number of degrees in the measure of one of the interior angles?

4. How many integer values of x make both of these inequalities true?

$$|2x-5| \geq 3 \quad \text{AND} \quad |x| \leq 7$$

5. What is the largest four-digit base 6 number that can be added to the base 6 number 5345_6 so that the sum will be a base 6 palindrome? (A palindrome is a number that reads the same forwards and backwards.)

Answers

1. _______________ $= A$
2. _____________$_7$ $= B_7$
3. _____________ $= C$
4. _____________ $= D$
5. _____________$_6$ $= E_6$
6. _________________

6. Using the values the team obtained in questions 1 through 5, find the value of the expression below. For B and E use the answers you wrote on the line as if they were base ten numbers, not base 6 or 7.

$$\sqrt{\frac{B-101D}{1000C}} + \sqrt{\frac{E-11}{A}}$$

Solutions to Category 6
Team Questions
Meet #3, January 2008

Answers

1. 19

2. 501313_7

3. 20

4. 13

5. 5502_6

6. 22

1. 16 + 49 = 65 which is one more than 64, so a triangle with sides 4, 7, and 8 would fit the definition of a "not quite" Pythagorean Triangle and the perimeter of the triangle would be 4 + 7 + 8 = **19**. There are other sets of three integers that fit the some of the requirements, such as 1, 2, 2 or 5, 5, 7. However neither is scalene.

2. Dividing 84535 by 7 gives us 12076 remainder 3.
Now dividing 12076 by 7 gives us 1725 remainder 1.
Now dividing 1725 by 7 gives us 246 remainder 3.
Now dividing 246 by 7 gives us 35 remainder 1.
Now dividing 35 by 7 gives us 5 remainder 0.
Now dividing 5 by 7 gives us 0 remainder 5.
The base 7 value is made up of the remainders : $\mathbf{501313_7}$.

3. By checking polygons with a small number of sides like 5 you find that the number of diagonals is not close to the number of degrees in each interior angle. So by jumping up to polygons with a larger number of sides the two values start getting closer. When the number of sides = **20**, the number of diagonals(170) finally grows larger than the number of degrees in the interior angle(162). Since you didn't have a calculator for this, checking just the values of *s* that yield whole number angle measures(15, 18, 20→factors of 360) would have been the best place to start.

Sides	diagonals	Interior angle
5	5	108
15	90	156
16	104	≈ 158
17	119	≈ 159
18	135	160
19	152	≈ 161
20	170	162

4. Solving the two inequalities we get

$\lvert 2x-5\rvert \geq 3$	AND	$\lvert x \leq 7 \rvert$
$2x-5 \geq 3$ or $2x-5 \leq -3$	AND	$x \leq 7$ and $x \geq -7$
$2x \geq 8$ or $2x \leq 2$	AND	$x \leq 7$ and $x \geq -7$
$x \geq 4$ or $x \leq 1$	AND	$x \leq 7$ and $x \geq -7$

The integers that fit both of those systems of inequalities are :

$-7, -6, -5, -4, -3, -2, -1, 0, 1$ and $4, 5, 6, 7$

For a total of **13** values.

5. Since we want the largest 4 digit base 6 number, we will start by trying to add 5000_6 to 5345_6 which would give us 14345_6. Since the 5 digit sum will start with a 1 we need it to end with a 1. We achieve that by adding 5002_6 instead to get 14351_6. We now have the outer digits the same in our would be palindrome. Remembering we want to add the largest possible 4 digit base 6 number, we try adding 5502_6 which gives us 15251_6 which is in fact a palindrome. If we try making 5502_6 any bigger though we would end up causing a carrying situation would would force a larger place value to be a smaller digit making the 4 digit number smaller, so $\mathbf{5502_6}$ is the largest we could add.

6. $\sqrt{\frac{B-101D}{1000C}} + \sqrt{\frac{E-11}{A}} = \sqrt{\frac{501313-101(13)}{1000(20)}} + \sqrt{\frac{5502-11}{19}} =$
$\sqrt{\frac{501313-1313}{20000}} + \sqrt{\frac{5491}{19}} = \sqrt{\frac{500000}{20000}} + \sqrt{289} = \sqrt{25} + \sqrt{289} = 5+17$

$=$ **22**