Meet #5
March 2007

Intermediate
Mathematics League
of
Eastern Massachusetts

Scores on a Percentage of Maximum Basis (100=Easiest):

	Q1	Q2	Q3	Q4	Q5	Q6
1 Mystery	97	89	31			
2 Geometry	50	36	53			
3 Number Th	44	83	14			
4 Arithmetic	81	56	89			
5 Algebra	78	86	42			
6 Team	83	100	67	67	67	50

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Category 1
Mystery
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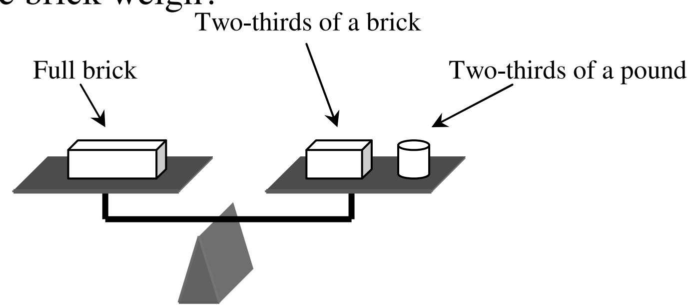
You may use a calculator.

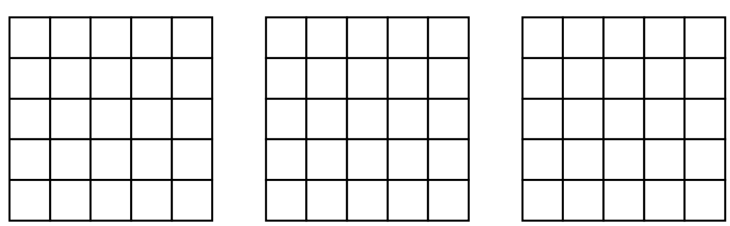
1. A brick balances perfectly with two thirds of a brick and two thirds of a pound. How many pounds does the brick weigh?



2. A card is selected from an ordinary deck. Aces count as 1, number cards as their face values, Jacks as 11, Queens as 12, and Kings as 13. The value of the card is doubled, then 1 is added, and then this result is multiplied by 5. If the selected card is a club, 1 is added; if it is a diamond, 2 is added; if it is a heart, 3 is added; and if it is a spade, 4 is added. If the final value for the selected card is 98, what card was selected? Name the card and the suit.

3. There are $2^{25} = 33{,}554{,}432$ different ways the unit squares on a 5-by-5 grid can be shaded black or left white. How many of these patterns have 90-degree rotational symmetry? Several grids are provided below for your exploration of this question. Note: A pattern has 90-degree rotational symmetry if it looks the same when you rotate it 90 degrees.

Answers
1. ____________________
2. ________ of ________
3. ____________________



Solutions to Category 1
Mystery
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Answers

1. 2

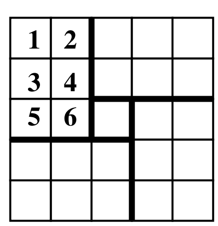
2. nine of hearts
or
9 of hearts

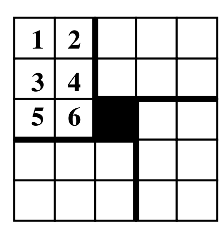
3. 128

1. Each third of the brick must weigh two thirds of a pound, so the brick must weigh $\frac{2}{3} \times 3 = \mathbf{2}$ pounds.

2. Multiples of 5 must have a 0 or a 5 in the ones place, so the card selected can only be a heart. Before the value of the suit was added, the value of the card must have been 98 – 3 = 95. Now we work backwards. Dividing by 5, we get 19. Subtracting 1, we get 18. Dividing by 2, we get 9. The card must have been a **nine (9) of hearts**.

3. First we partition the 5-by-5 grid as shown below. Now it is clear that we will have to repeat any pattern of black and white squares we make in the six numbered squares if we want to have 90-degree rotational symmetry. Each of the six squares can be either black or white, so there are $2^6 = 64$ possible patterns with a *white* square in the middle and $2^6 = 64$ possible patterns with a *black* square in the middle. Therefore there are 64 + 64 = **128**, or $2^7 =$ **128**, patterns that have 90-degree rotational symmetry.





Category 2
Geometry
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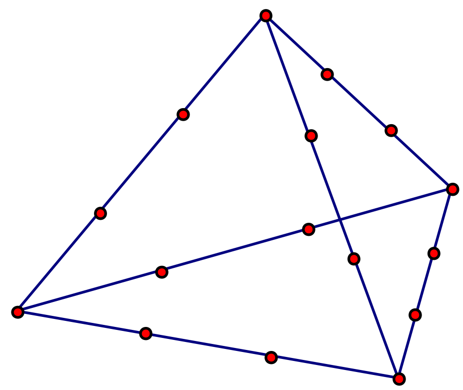
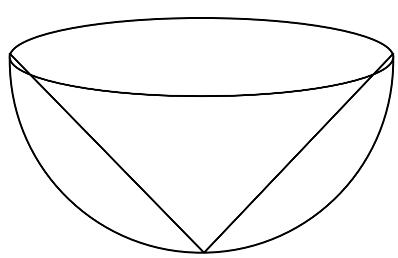
You may use a calculator.

1. Each edge of the tetrahedron shown at right has been trisected. Suppose a tetrahedron with one-third the edge length is cut from *each* vertex of the tetrahedron, using these points to guide the cutting. How many edges will there be on the resulting solid?



2. Two teams, the Stackers and the Packers, were each given 1729 unit cubes and told to arrange all the cubes to make just two cubes. It turns out that the two teams found different solutions to this problem. How many square units are in the surface area of the largest of the four cubes that were made?

3. The figure below depicts a cone inside a hemisphere. If the diameters of both the cone and the hemisphere are 6 centimeters and the height of the cone is 3 centimeters, how many cubic centimeters are in the hemisphere but not in the cone? Use 3.14 for π and express your answer to the nearest hundredth of a cubic centimeter.



Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 2
Geometry
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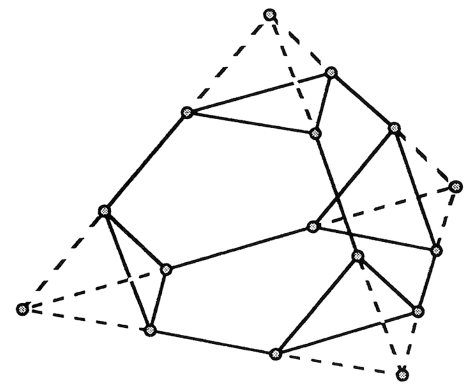
Answers

1. 18

2. 864

3. 28.26

1. In place of each of the four vertices, we will get three new edges. There will be $3 \times 4 = 12$ new edges, in addition to the original 6 edges for a total of **18** edges.



2. Suppose the Stackers make a 10 by 10 by 10 cube, which uses 1000 unit cubes, and a 9 by 9 by 9 cube, which uses 729 unit cubes. Then the Packers must have made a 12 by 12 by 12 cube, which uses 1728 unit cubes, and a 1 by 1 by 1 cube, which uses 1 cube. The surface area of the largest cube is $6 \times 12^2 = 6 \times 144 = \mathbf{864}$ square units.

3. The volume of a sphere is given by the formula $V_{\text{Sphere}} = \frac{4}{3}\pi r^3$. The volume of a hemisphere would be half that amount. The radius of our hemisphere is 3 cm, so the volume is $\frac{4}{3} \times \pi \times 3^3 \div 2 = 18\pi$. The volume of a cone is given by the formula $V_{Cone} = \frac{1}{3}\pi r^2 h$. The radius and the height of our cone are both 3 cm, so the volume is $\frac{1}{3} \times \pi \times 3^2 \times 3 = 9\pi$.
The difference between the two volumes is $18\pi - 9\pi = 9\pi \approx 9 \times 3.14 =$ **28.26** cubic centimeters.

Category 3
Number Theory
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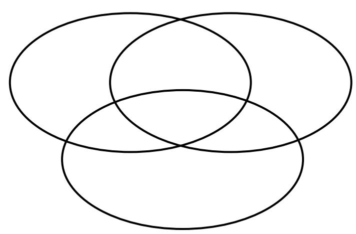
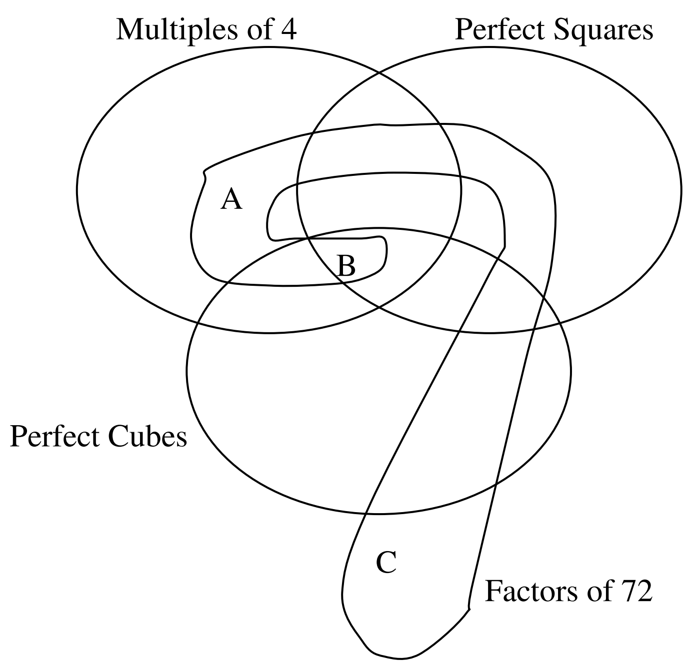
You may use a calculator.

1. Set A is all the positive whole number multiples of 13 that are less than 100. Set B is all the positive whole numbers that are one more than a multiple of 5 and less than 100. How many numbers are there in $A \cup B$? Reminder: $\cup$ means union. Also, we will include the number 1 in set B.

2. A guessing game has 24 cards with pictures of different people. Five people in the pictures have white hair, five people wear glasses, and 10 people wear hats. Two of the people with white hair wear glasses and two of the people with white hair wear hats. Nobody who wears glasses wears a hat. How many of the 24 different people do not have white hair or wear glasses or wear a hat?



3. Venn diagrams work well for two or three sets, but not as well for four or more sets. The noodle-shaped region below shows one way that a fourth set might be included in a Venn diagram. If the natural numbers 1 through 72 inclusive are placed in the appropriate regions, what is the sum of the numbers in regions A, B, and C?



Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 3
Number Theory
Meet #5, March 2007

Answers

1. 25

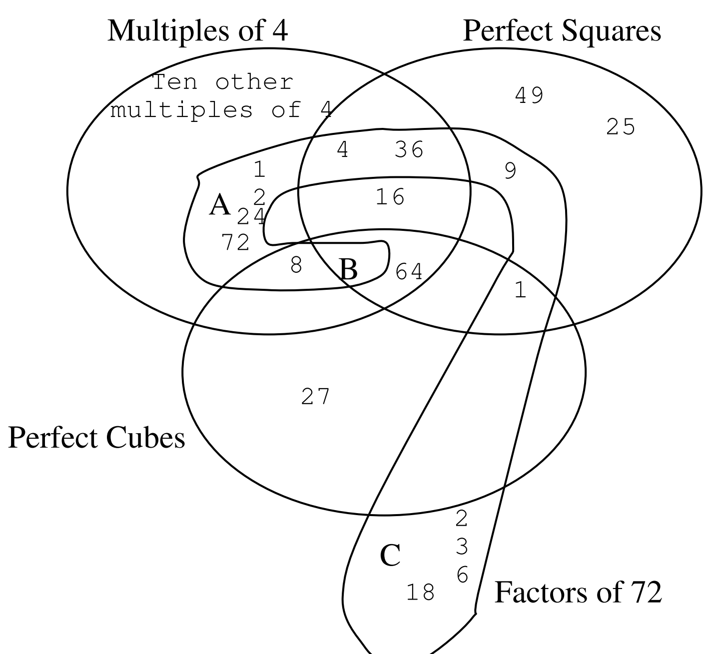
2. 8

3. 137

1. There are 7 multiples of 13 less than 100: 13, **26**, 39, 52, 65, 78, and **91**. There are 20 numbers less than 100 that are one more than a multiple of 5: 1, 6, 11, 16, 21, **26**, 31, 36, 41, 46, 51, 56, 61, 66, 71, 76, 81, 86, **91**, and 96. The two numbers that occur on both lists are in bold. Thus there are 7 + 20 – 2 = **25** numbers less than 100 that are either multiples of 13 or one more than a multiple of 5.

2. If the sets were mutually exclusive, there would be 5 + 5 + 10 = 20 people in our three sets. But two of the people with white hair wear glasses and two of the people with white hair wear a hat, so there must be only 20 – 2 – 2 = 16 people in our three sets. There are 24 people in the game, so there must be 24 – 16 = **8** people who do not have white hair or wear glasses or wear a hat.

3. We need only consider the factors of 72, but the diagram below shows all the numbers 1 through 72. Region A has 12, 24, and 72, which adds up to 108. There are no numbers in region B. Region C has 2, 3, 6, and 18, which adds up to 29. The desired total is 108 + 0 + 29 = **137**.



Category 4
Arithmetic
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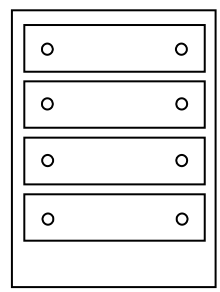
You may use a calculator.

1. At Motivation Middle School in Perfection, Pennsylvania, seven students got perfect scores on the MATHCOUNTS School Competition. How many ways can four students be chosen from these seven to form the school's MATHCOUNTS team for the Regional Competition?

2. As an April Fool's joke, Terry rearranged the drawers on his sister's bureau. He took all four drawers out and then put them back in different slots. As it turned out, he put exactly one of the drawers back in its original slot. How many different ways could Terry have done this?



3. There are red, purple, and white marbles in a jar. For every two red marbles there are five purple marbles. For every three purple marbles there are four white marbles. What is the least number of marbles that could be in the jar?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 4
Arithmetic
Meet #5, March 2007

Answers

1. 35

2. 8

3. 41

1. We calculate seven choose four as follows:

$$ {}_7C_4 = \frac{7!}{4! \cdot (7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \mathbf{35} $$

There are **35** different ways a team of 4 can be chosen from the 7 students.

2. For each of the four drawers that could be the one in its original slot, there are only two ways to arrange the other three drawers so that none of them are in their original slot. Thus there are $4 \times 2 = \mathbf{8}$ ways that Terry could have rearranged his sisters drawers.

3. Both of the other colors are compared to purple, so lets find the least common multiple for 5 and 3. That would be 15. If there are 15 purple, then there are 6 red and 20 white. The *least* number of marbles must be 15 + 6 + 20 = **41**, but there could be any multiple of 41 marbles in the jar.

Category 5
Algebra
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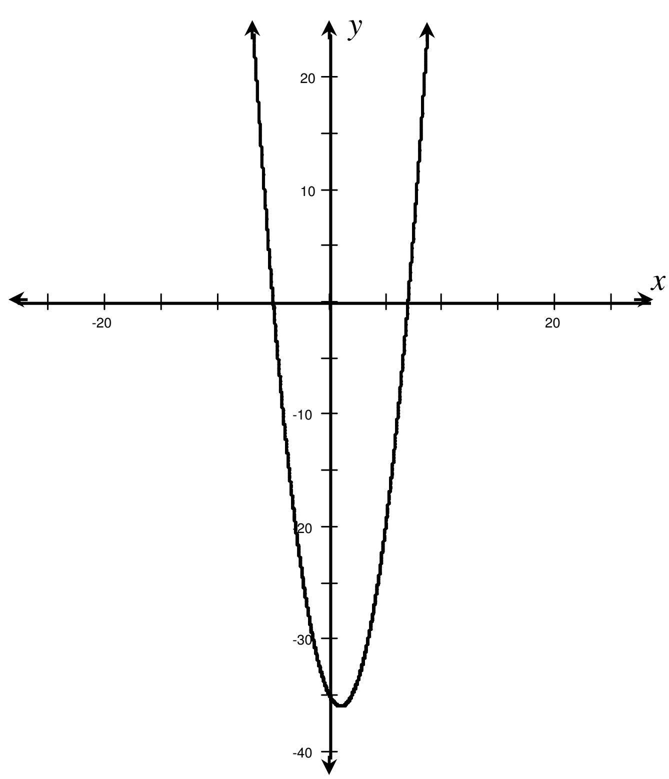
You may use a calculator.

1. Give the lesser of the two solutions to the following equation.

$$x(7+x)+312-4x=3(x+137)+526$$

2. The length and width of a certain rectangle are whole numbers of units, and the length is one more than four times the width. If the area of the rectangle is 105 square units, how many units are in the perimeter of the rectangle?

3. The graph of the qaudratic equation shown at right crosses the *x*-axis at the points (–5, 0) and (7, 0), and its vertex is at the point (1, –36). Give the coordinates of the two points where the graph of this same equation crosses the line $y = 13$. Reminder: The coordinates of the two points should be written as ordered pairs in parentheses.



Answers
1. ______________________
2. ______________________
3. _________ and _________

Solutions to Category 5
Algebra
Meet #5, March 2007

Answers

1. –25

2. 52

3. (8, 13) and (–6, 13)

1. The equation can be simplified as follows:

$$x(7+x)+312-4x=3(x+137)+526$$
$$7x+x^2+312-4x=3x+411+526$$
$$x^2+3x+312=3x+937$$
$$x^2+312=937$$
$$x^2=625$$

The two solutions are x = 25 and x = –25. We want **–25**.

2. If we call the width x, then the length is $4x + 1$. Since length times width gives the area of a rectangle, we can write the quadratic equation $x(4x+1)=105$. Distributing on the left, we can rewrite this as $4x^2+x=105$. Now we subtract 105 from each side to set the equation equal to zero: $4x^2+x-105=0$. At this point we can use the quadratic formula or try to factor the trinomial into a product of two binomials. Our template for factoring is: $(\_\_\,x+\_\_\_)(\_\_\,x-\_\_\_)=0$. The prime factorization of 4 is 2 × 2 and that of 105 is 3 × 5 × 7. We need to place these factors in the blanks so that the difference is 1 for the middle term. We note that 2 × 2 × 5 = 20 and 3 × 7 = 21, so our equation must be $(4x+21)(x-5)=0$. The two solutions are $x=-\frac{21}{4}$ and $x=5$. Only 5 units make sense for the width, so the length must be 4 × 5 + 1 = 21 units. The perimeter of the rectangle is 2 × (5 + 21) = 2 × 26 = **52** units.

3. We are told that the equation is quadratic and we are given the roots of the equation. We can assume that the equation has the form $y=a(x+5)(x-7)$, where a is not yet determined. We can use the vertex, (1, –36), as a point to help us determine the value of a. First we substitute y = -36 and x = 1 into the equation to get $-36=a(1+5)(1-7)$. Now we simplify and get $-36=-36a$, which shows us that a = 1. Now we want to find the values of x when y = 13. We have to solve the equation $13=1(x+5)(x-7)$. Expanding on the right, we get $13=x^2-2x-35$. Now we set this equal to zero and factor the new equation as follows: $0=x^2-2x-48=(x-8)(x+6)$. From this, we find that the desired x values are 8 and –6. The coordinates of the two points where the graph crosses the line y = 13 are **(8, 13)** and **(–6, 13)**.

Category 6 *You may use a calculator.*
Team Questions
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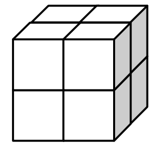
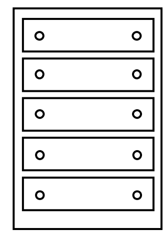


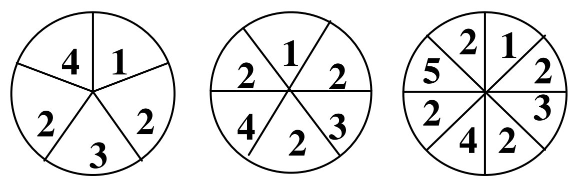
1. How many different line segments are there that connect the centers of different unit cubes in a 2-by-2-by-2 cube?

2. In a small company, 13 people earn \$30,000, 4 people earn \$40,000, 2 people earn \$50,000, and 1 person earns \$200,000 a year. What percent of the employees earn less than the average salary?



3. As an April Fool's joke, Terry rearranged the drawers on his *brother's* bureau. He took all *five* drawers out and then put them back in different slots. As it turned out, he put exactly one of the drawers back in its original slot. How many different ways could Terry have done this?

4. John gets to pick one of the spinners shown below, spin the spinner twice, and then add the two numbers. If he gets an even sum, he wins a new calculator. If he chooses the spinner with the highest likelihood of success, what is the probability of his getting an even sum? Express your answer as a fraction in lowest terms. Note: In each spinner, all regions are equal.



Spinner A Spinner B Spinner C

5. Each stick in a box has a whole number side length. No selection of three sticks from the box will form a triangle. (In other words, any selection of three sticks from the bag will *fail* to form a triangle.) If the longest stick in the bag is less than 80 cm long, what is the maximum number of sticks in the bag?

Answers
1. ______________ = A
2. ______________ = B
3. ______________ = C
4. ______________ = D
5. ______________ = E
6. ________________

6. Using the values the team obtained in questions 1 through 5, evaluate the equation below. For B you should use the decimal or the fraction that is equivalent to the percent you gave.

$$\frac{8A + \dfrac{C+E}{D}}{B}$$

Solutions to Category 6
Team Questions
Meet #5, March 2007

Answers

1. 28

2. 85 or 85%

3. 45

4. $\frac{5}{9}$

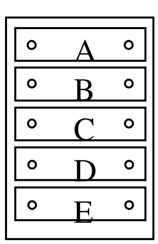
5. 10

6. 380

1. The centers of each of the eight cubes can be connected to the centers of all seven of the other cubes. If we multiply 7×8 and get 56, we have counted every line segment at both ends. Therefore there must be only $56 \div 2 =$ **28** line segments connecting centers of unit cubes in a 2-by-2-by-2 cube.

2. There are a total of $13 + 4 + 2 + 1 = 20$ people in the company. The total of their salaries, in thousands of dollars, is $13 \times 30 + 4 \times 40 + 2 \times 50 + 200 = 390 + 160 + 100 + 200 = 850$. Dividing 850 by 20, we get 42.5, which means the average salary is $42,500. Seventeen of the twenty employees earn less than this average. Dividing 17 by 20 and multiplying by 100, we find that this is **85%** of the employees.

3. First let's label the drawers with the letters A through E as shown at left. Any one of the five drawers could be the one drawer that goes back to its original slot. Let's say it's drawer A. The tricky part is to find out how many ways the other four drawers can be arranged so that *none* of them is in its original slot. There are $4! = 4 \times 3 \times 2 \times 1 = 24$ different ways to arrange four different items, such as drawers or letters. Of these, the following nine (9) arrangements have none of the letters in its original spot.



A	A	A	A	A	A	A	A	A
C	C	C	D	D	D	E	E	E
B	D	E	B	E	E	B	D	D
E	E	B	E	B	C	C	B	C
D	B	D	C	C	B	D	C	B

Since it could be any of the five drawers that is in its original slot, there are $5 \times 9 =$ **45** different ways that Terry could have rearranged the drawers of his brother's dresser.

4. There are two ways to get an even sum: an even plus an even or an odd plus an odd. The probability of getting an even sum for each spinner is given below.

For Spinner A: $\frac{3}{5}\times\frac{3}{5}+\frac{2}{5}\times\frac{2}{5}=\frac{9}{25}+\frac{4}{25}=\frac{13}{25}$

For spinner B: $\frac{4}{6}\times\frac{4}{6}+\frac{2}{6}\times\frac{2}{6}=\frac{16}{36}+\frac{4}{36}=\frac{20}{36}=\frac{5}{9}$

For spinner C: $\frac{5}{8}\times\frac{5}{8}+\frac{3}{8}\times\frac{3}{8}=\frac{25}{64}+\frac{9}{64}=\frac{34}{64}$

The least common denominator for 25, 9, and 64 is 14,400. Comparing these fractions, we have $\frac{13}{25}=0.52$, $\frac{5}{9}=0.\overline{5}$, and $\frac{34}{64}=0.53125$. Thus Spinner B is the most likely to give an even sum, and the probability is $\mathbf{\frac{5}{9}}$.

5. If we start at 80 and work our way down, we might get a sequence of possible lengths such as 80, 40, 20, 10, 5, 3, 2, 1, 1. Note that we can have two sticks of length 1 cm. On the other hand, if we start with two sticks of length 1 cm, we will get the familiar Fibonacci Sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. Notice that our first list had only nine numbers, whereas our second list had **10** numbers, which is the maximum number of sticks in the bag.

6. Substituting the correct values into the expression, we get:

$$\frac{8A+\frac{C+E}{D}}{B}=\frac{8\times28+\frac{45+10}{\frac{5}{9}}}{\frac{17}{20}}=\frac{224+\frac{55}{\frac{5}{9}}}{\frac{17}{20}}=\frac{224+99}{\frac{17}{20}}=323\times\frac{20}{17}=\mathbf{380}$$