Meet #4
February 2007

Intermediate Mathematics League of Eastern Massachusetts

Relative scores on questions: 100 = highest scoring question (based on one cluster)						
	Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**
1 Mystery	**100**	81	**97**			
2 Geometry	84	81	69			
3 Number Theory	72	81	81			
4 Arithmetic	**91**	63	**44**			
5 Algebra	53	59	**41**			
6 Team	56	38	47	47	28	28

League Average Scores

Round	1	2	3	4	5	6	**Total**
Score	25.7	12.8	14.9	11.2	10.3	16.0	**91.1**

Meet #4
February 2007

Category 1 *You may use a calculator today!*
Mystery
Meet #4, February 2007

1. The ratio of boys to girls in Lindsey's gym class is 5 to 6. There are four more girls than boys in the class. How many students are there in Lindsey's gym class?

2. The temperature at noon is recorded each day for five days in a row. If the temperatures are all different integers and their product is 12, what was the temperature on the warmest day?

3. The graph below right shows the number of male and female cousins for five different people. Use the clues to find out which names go with which letters. Then write the names in the appropriate spaces below.

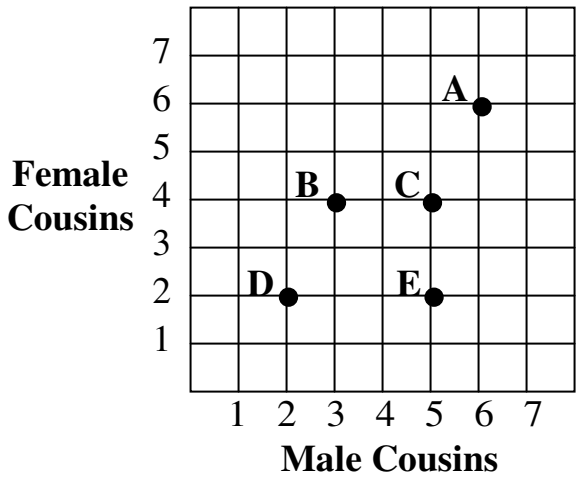
Clues:
- Gus and Kurt have the same number of female cousins.
- Fred has two more male cousins than Louisa.
- Kurt has the same number of male cousins as female cousins.
- Martha likes to do crossword puzzles.

Answers

1. _____

2. _____ degrees

3. A:_____
   B:_____
   C:_____
   D:_____
   E:_____



Solutions to Category 1
Mystery
Meet #4, February 2007

Answers

1. 44

2. 3

3. A: Martha
B: Louisa
C: Fred
D: Kurt
E: Gus

1. There must be $4 \times 5 = 20$ boys and $4 \times 6 = 24$ girls, for a total of $20 + 24 =$ **44** students in the gym class.

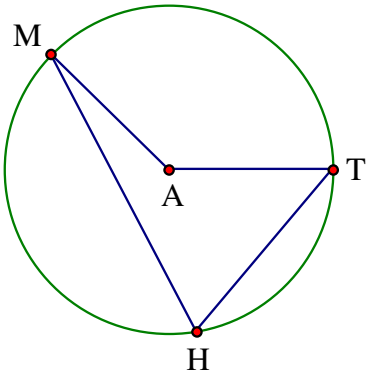
2. The prime factorization of 12 is $2 \times 2 \times 3$. If we wish to find five different integers with a product of 12, we will have to use two negative and also the multiplicative identity 1. The numbers are –1, –2, 1, 2, and 3. The temperature on the warmest day must have been 3 degrees.

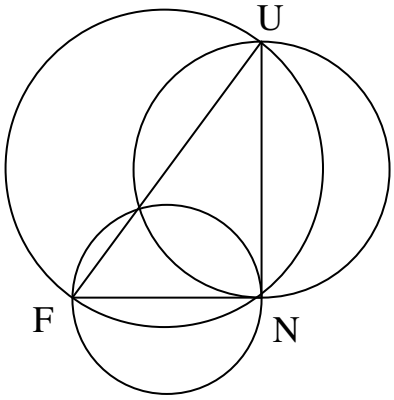
3. The first clue tells us that Gus and Kurt are either B and C or D and E. Since five is two more than three, the next clue tells us that Louisa must be B. Fred must be C, not E, since Gus and Kurt will have to be D and E. The third clue tells us that Kurt is D and Gus must be E. This leaves Martha to be A. The order of the names from A to E is **Martha, Louisa, Fred, Kurt, Gus**. (All names must be correct for 2 points.)

Category 2
Geometry
Meet #4, February 2007

You may use a calculator today!

1. In the figure at right, A is the center of the circle and M, T, and H are points on the circle.
If the measure of angle MAT is 136 degrees, how many degrees are in the measure of angle MHT?



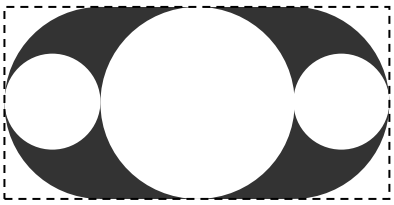


2. Right triangle FUN has side lengths 3 units, 4 units, and 5 units. These lengths are the diameters of the three circles that are drawn on each side. How many units are in the curved perimeter of the figure? (By perimeter, we mean the outer-most path around the entire figure.) Use 3.1416 for π and round your answer to the nearest hundredth of a unit.

3. The figure below started out as a rectangle that measured 4 feet by 8 feet. The corners were then trimmed to be semicircles with diameter 4 feet. Two circles with diameter 2 feet and one circle with diameter 4 feet were also cut out, as shown. How many square feet are in the area of the shaded figure that remains? Express your answer to the nearest tenth of a square foot.

Answers

1. _____
2. _____
3. _____



Solutions to Category 2
Geometry
Meet #4, February 2007

Answers

1. 68

2. 18.85

3. 9.7

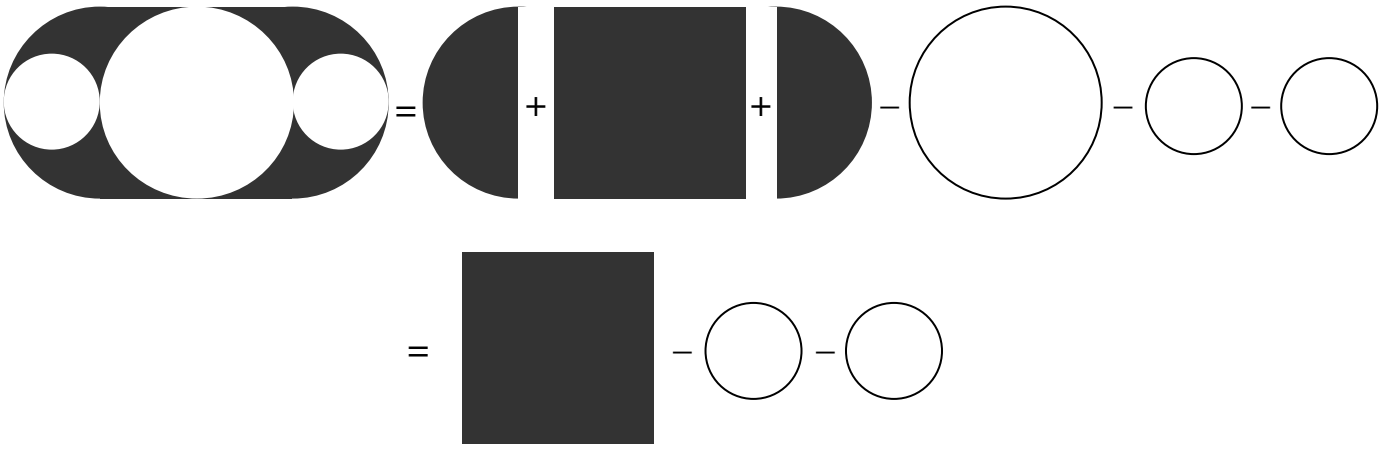
1. Angle MAT is known as a central angle and angle MHT is known as an inscribed angle. If a central angle and an inscribed angle both intercept the same arc of the circle (arc MT in this case), then the measure of the inscribed angle is exactly half the measure of the central angle.

2. The curved perimeter of the figure is the sum of the three semicircles, and can be calculated as follows:

$$\frac{3\pi}{2}+\frac{4\pi}{2}+\frac{5\pi}{2}=(3+4+5)\frac{\pi}{2}=12\cdot\frac{\pi}{2}=6\pi\approx 6\cdot 3.1416=18.8496$$

Rounding this value to the nearest hundredth of a unit, we get **18.85** units.

3. The figure can be thought of as a square and two large semicircles with a large circle and two small circles removed. The area of the two large semicircles is negated by the large circle that is removed. Each small circle has a diameter of two feet, which means a radius of 1 foot and an area of π square feet. The remaining area is equivalent to a 4-by-4 square minus two unit circles, which is 16 – 2π, or about **9.7** square feet to the nearest tenth of a square foot.



Category 3
Number Theory
Meet #4, February 2007

You may use a calculator today!

1. A Ferris wheel has 36 chairs numbered 1 through 36 in a loop. The operator loads chair #11 first and then every 11^{th} chair after that. If Almonzo and Royal are on the 11^{th} chair that gets loaded, what is the number on the chair they are riding in?

2. The sum of the first 11 terms in an arithmetic sequence is 693. If the constant difference between consecutive terms is 11, what is the first term of the sequence?

3. Find the sum of the values of the digits A, B, and C in the four terms of the increasing arithmetic sequence below.

$$AB8,\ A65,\ C0B,\ CA9$$

Answers

1. _____
2. _____
3. _____

Solutions to Category 3
Number Theory
Meet #4, February 2007

Answers

1. 13

2. 8

3. 9

1. The order in which the chairs are loaded is as follows: 11, 22, 33, 8, 19, 30, 5, 16, 27, 2, **13**. Almonzo and Royal are riding in chair #**13**. We can also calculate this as $11 \times 11 = 121$. 108 is a multiple of 36, so $121 - 108 = 13$.

2. Let's call the first term x. Then the sequence looks like this:

$$x + (x+11) + (x+22) + (x+33) + ... + (x+110) = 693$$

This can be rewritten as $11x + 11 + 22 + 33 + ... + 110 = 693$. Let's simplify by dividing both sides of the equation by 11.

$x + 1 + 2 + 3 + ... + 10 = 63$ The sum of the numbers from 1 to 10 is 55, so x must be $63 - 55 =$ **8**.

3. An arithmetic sequence has a constant difference between the terms. Comparing the units digits of $AB8$ and $A65$, we can see that the units digit of the constant difference must be a 7. Comparing $A65$ and $C0B$, we see that the constant difference must be 37. Adding 65 + 37, we get 102, so $B = 2$. 102 + 37 = 139, so $A = 3$. The four terms of the sequence must be 328, 365, 402, 439. The desired sum is $A + B + C = 3 + 2 + 4 =$ **9**.

Category 4
Arithmetic
Meet #4, February 2007

You may use a calculator today!

1. If you increase Meghan's favorite fraction by 25% you get Eva's favorite fraction. The difference between their favorite fractions is $\frac{1}{12}$. What is Meghan's favorite fraction if she always expresses it in lowest terms?

2. The dinner bill for the party of nine people came to $240.65, before the sales tax was added. Jim calculated the amount of a 15% tip on the amount *before* the 5% sales tax was added. Craig calculated the amount of a 15% tip on the amount *after* the 5% sales tax was added. How much more would the waiter get if the party agreed to pay Craig's tip instead of Jim's tip? Express your answer in dollars to the nearest hundredth of a dollar.

3. A bank plans to offer a savings account at 4.8% simple annual interest. If the bank compounds monthly, what is the annual yield for this savings account? (The annual yield is the amount of interest a given balance would earn when the effect of the compounding interest is included.) Do not round until the last step, and give your answer as a percent to the nearest tenth of a percent.

Answers

1. _____
2. _____
3. _____

Solutions to Category 4
Arithmetic
Meet #4, February 2007

Answers

1. $\frac{1}{3}$

2. \$1.80

3. 4.9

1. Let's call Meghan's favorite fraction x. Then Eva's favorite fraction is x plus 25% of x. This can be written as $1.25x$ or $\frac{5}{4}x$. The positive difference between x and $\frac{5}{4}x$ is $\frac{5}{4}x - x = \frac{1}{4}x$. If $\frac{1}{4}x = \frac{1}{12}$, then $\frac{4}{4}x = \frac{4}{12}$. Meghan's favorite fraction must be $\mathbf{\frac{1}{3}}$, since she always expresses it in lowest terms.

2. The tip as calculated by Jim would be $\$240.65 \times 0.15 = \$36.0975$ or about \$36.10. The dinner bill after tax comes to $240.65 \times 1.05 = \$252.6825$ or about \$252.68. The tip as calculated by Craig would be $\$252.68 \times 0.15 = \$37.90$. The difference between these two amounts is \$37.90 – \$36.10 = \$**1.80**. In general, Craig's method of calculating the 15% tip on the *after tax total* amounts to a slightly more generous tip of $1.05 \times 0.15 = 0.1575 = 15.75\%$.

3. With an annual interest rate of 4.8%, the monthly interest rate would be $4.8\% \div 12 = 0.4\%$. As a decimal, 0.4% is 0.004, or 4 thousandths. Since the interest on the balance is added to the balance each month, the easiest way to calculate it is to simply multiply by 1.004. If we multiply the original amount by 1.004 each month for 12 months, the compounded value will be 1.004^{12}, which is about 1.049 times as much. The annual yield is the extra 0.049, or **4.9%**.

Category 5
Algebra
Meet #4, February 2007

You may use a calculator today!

1. On her most recent birthday, Emily became $\frac{1}{3}$ of her father's age. In six years, she will be $\frac{5}{12}$ of her father's age. How old was Emily's father when she was born?

2. Dennis usually averages 70 miles per hour on his trip to his grandmother's house. When he was towing a trailer he averaged only 60 miles per hour, and the trip took half an hour longer than usual. How many miles is it to his grandmother's house?

3. In a warehouse there are three kinds of boxes.
If you stack two A boxes, one B box, and two C boxes, the stack is 88 inches tall. If you stack one A box, two B boxes, and one C box, the stack is 77 inches tall. If you stack three A boxes, one B box, and one C box, the stack is 83 inches tall. If you make a stack of C boxes that is as close as possible to eight feet tall, how many inches less than eight feet tall is the stack?

Answers

1. _____
2. _____
3. _____

Solutions to Category 5
Algebra
Meet #4, February 2007

Answers

1. 28

2. 210

3. 1

1. Let Emily's father's age now be x years. Then Emily's age now is $\frac{1}{3}x$ years. In six years, Emily's father will be $x + 6$. We can calculate Emily's age six years from now as $\frac{1}{3}x + 6$ and as $\frac{5}{12}(x + 6)$. These are, of course, equal, so we can write the following equation and solve for x: $\frac{1}{3}x + 6 = \frac{5}{12}(x + 6)$. Multiplying both sides by 12, we get $4x + 72 = 5(x + 6)$. Now we can distribute on the right, which give us $4x + 72 = 5x + 30$. Finally, subtracting $4x$ and 30 from both sides, we get $x = 42$. That's Emily's fathers age now. Emily must be $\frac{1}{3} \cdot 42 = 14$ now, so her father must have been 42 – 14 = **28** years old when she was born.

2. Let the distance to Dennis's grandmother's house be d miles. If Dennis travels at 60 mph instead of 70 mph, he is traveling $\frac{6}{7}$ of his normal speed, so it will take $\frac{7}{6}$ the amount of time. We know that the extra sixth of the time is half an hour. This means it usually takes six half hours, or three hours. It must be 3 hrs × 70 mph = **210** miles to Dennis's grandmother's house.

3. From the information given, we can write the following system of equations.

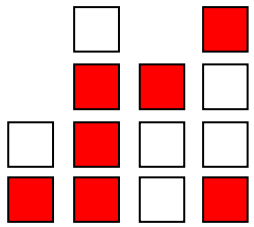
$$\begin{cases} 2A + B + 2C = 88 \\ A + 2B + C = 77 \\ 3A + B + C = 83 \end{cases}$$

If we double the second equation and subtract the first, we get $3B = 66$, so $B = 22$. Subtracting the second equation from the third, we get $2A - B = 6$. If we substitute $B = 22$ into this equation, we get $2A - 22 = 6$. Adding 22 to both sides, we find that $2A = 28$, so $A = 14$. Now we can solve any of the three equations for C and find that $C = 19$. If we stack 5 of these 19-inch boxes, our stack would be 95 inches tall, which is **1** inch less than eight feet.

Category 6
Team Questions
Meet #4, February 2007

You may use a calculator today!

1. In a binary clock, the shaded squares in each column give the digits for the corresponding military time in hours and minutes. What time is given by the binary clock at right? Note: Military time goes to 24:00 hours.



__ __ : __ __

2. Some of the factors of a number are 9, 15, and 21. If the number has exactly nine more factors, what is the number?

3. The interior angles of an irregular pentagon form an arithmetic sequence. The measure of one of the angles is 80°. Find the sum of all possible values of the largest angle.

4. Find the sum of the series $1(1!)+2(2!)+3(3!)+...+10(10!)$.

5. Robert has a 100-foot roll of wire fencing. He wants to create a dog pen in the shape of a golden rectangle. A golden rectangle has a length that is about 1.618 times its width. If Robert rounds to the nearest whole number of feet for the length and width of his pen and the 100 feet of wire fencing is the entire perimeter, how many square feet will there be in the dog pen?

6. Ignore the colon in the answer to question 1 and let F equal the largest prime factor of the four-digit number A.
Let G equal the greatest common factor of B and C.
Let H equal the last three digits of D.
Let I equal the nearest whole number to $\sqrt{E}$.
Now evaluate the following expression:

$$FG - H + I$$

Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____ = E
6. _____

Solutions to Category 6
Team Questions
Meet #4, February 2007

Answers

1. 17:49

2. 315

3. 300

4. 39,916,799

5. 589

6. 20

1. The bottom square in each column is the 1 square. The next square up is the 2 square, the square after that is the 4 square, and the top square is the 8 square. The sum of these values gives the digit for each place value of the military time. The first column is just 1; the second column is 1 + 2 + 4 = 7; the third column is just 4; and the fourth column is 1 + 8 = 9. The time shown is thus **17:49**.

2. Let's consider the prime factors of the three factors that are known. $9 = 3 \times 3$, $15 = 3 \times 5$, and $21 = 3 \times 7$. The unknown number must be at least $3 \times 3 \times 5 \times 7 = 315$. In fact, 315 does have nine more factors, so the number must be **315** and not some larger multiple of 315. The twelve factors of 315 are:

1	5	7	35
3	15	21	105
9	45	63	315

3. A pentagon may be divided into three triangles, each with an angle sum of 180 degrees. That makes a total of 540 degrees for the five interior angles. On a regular pentagon, they would all measure $540 \div 5 = 108$ degrees. On our pentagon, however, the angle measures form an arithmetic sequence. It follows that one of the angles must be 108 degrees. The five angles must be: $108 - 2d$, $108 - d$, 108, $108 + d$, and $108 + 2d$, where d is the constant difference between the terms in the arithmetic sequence. Since one of the other angles is 80 degrees, then the difference between 108 and 80 is either the constant difference, d, or twice the constant difference $2d$. The five angle measures are either 52, 80, 108, 136, and 164, or they are 80, 94, 108, 122, and 136. The two possible values of the largest angle are 164 and 136, so the desired sum is 164 + 136 = **300**.

4. Many calculators have a factorial key. The calculation is given below.

$1(1!)+2(2!)+3(3!)+4(4!)+5(5!)+6(6!)+7(7!)+8(8!)+9(9!)+10(10!)$

$=1\cdot 1+2\cdot 2+3\cdot 6+4\cdot 24+5\cdot 120+6\cdot 720+7\cdot 5040+8\cdot 40320+9\cdot 362880+10\cdot 3628800$

$=1+4+18+96+600+4320+35280+322560+3265920+36288000$

$=39916799$ or **39,916,799**

Interestingly, this is equivalent to 11! – 1. You might check to see if this is true in general.

5. Since the 100 feet of wire fencing must make two lengths and two widths, let's just focus on the 50 feet that will make up one length and one width. We are looking for a width, x, and a length that is $1.618x$, such that $x + 1.618x = 50$ feet. This simplifies to $2.618x = 50$. Dividing both sides of the equation by 2.618, we get $x = 19.098$ feet, or 19 feet to the nearest whole number. That means the length is 31 feet, and the area of the dog pen is $19 \times 31 =$ **589** square feet.

6. The prime factorization of 1749 is $3 \times 11 \times 53$, so $F = 53$. The greatest common factor of 315 and 300 is 15, so $G = 15$. The last three digits of 39,916,799 are 799, so $H = 799$. The square root of 589 is about 24.27, so $I = 24$. The value of $FG - H + I$ is thus $53 \times 15 - 799 + 24 = 795 - 799 + 24 =$ **20**.