Meet #3
January 2007

Intermediate Mathematics League of Eastern Massachusetts

Relative scores on questions: 100 = max, 0 = min (based on one cluster)						
	Q1	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**
1 Mystery	**100**	61	39			
2 Geometry	86	89	89			
3 Number Theory	50	58	**31**			
4 Arithmetic	83	58	53			
5 Algebra	69	64	**33**			
6 Team	83	100	83	100	67	83

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Category 1
Mystery
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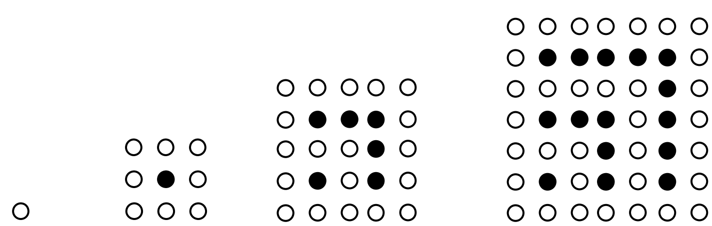
1. The Flingers beat the Lobbers by 6 points in the basketball game. If the total number of points scored was 96, how many points did the Flingers score?

2. Place the numbers 1 through 9 in the squares, using each number once. The numbers in the circles are the sums of the rows and the columns. The phrases in the squares give clues about the numbers in those squares. What is the sum of the three numbers on the diagonal from the upper left to the lower right?

Power of 3	Power of 3		17
Power of 2	Power of 3	Power of 2	13
	Factor of 6	Power of 2	15
8	18	19	

3. Suppose the pattern below were continued. How many more white dots than black dots would there be in the seventh stage?



Stage 1 Stage 2 Stage 3 Stage 4

Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 1
Mystery
Meet #3, January 2007

Answers

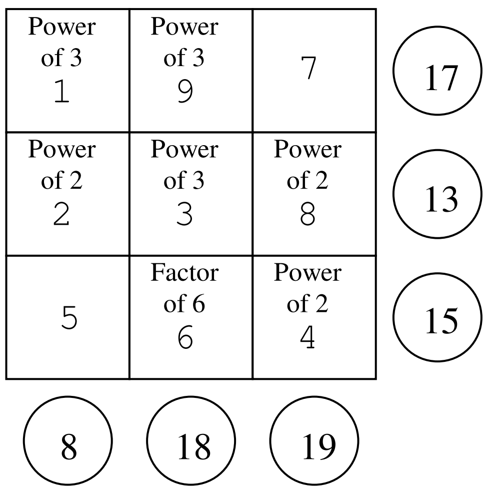
1. 51

2. 8

3. 37

1. Suppose the Lobbers had scored another 6 points. Then the Flingers and the Lobbers would have tied, with a total of 96 + 6 = 102 points. The Flingers must have scored 102 ÷ 2 = **51** points. (The Lobbers actually scored 51 – 6 = 45.)

2. Since all three of the single-digit powers of 3 (1, 3, and 9) are used, the three single-digit powers of 2 must be 2, 4, and 8. The only factor of 6 that remains is 6 itself, so we can fill in that square. The two squares without a clue must be 5 and 7, and the 7 won't allow a sum of 8 in the first column. This means the bottom left square must be 5. Now the only possible sum for the first column is 1 + 2 + 5 = 8. Next we see that the bottom right square must be 4, and the remaining numbers are easily placed. The sum of the three numbers on the diagonal from the upper left to the lower right is 1 + 3 + 4 = **8**.



3. The total number of dots at each stage is an odd square number. The number of white dots increases by 7, then 11, then 15—always increasing by four more than it increased the last time. The same is true of black. The difference between the number of white dots and the number of black dots is increasing by 6 at each stage. This difference can be calculated as six times the stage number minus five. At the seventh stage, there will be $6 \times 7 - 5 = 42 - 5 = \mathbf{37}$.

Stage	**Whites**	**Blacks**	**Total**	**Difference**
1	1	0	1	1
2	8	1	9	7
3	19	6	25	13
4	34	15	49	19
5	53	28	81	25
6	76	45	121	31
7	103	66	169	**37**

Category 2
Geometry
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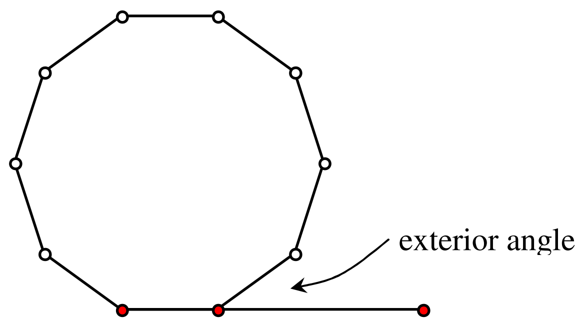
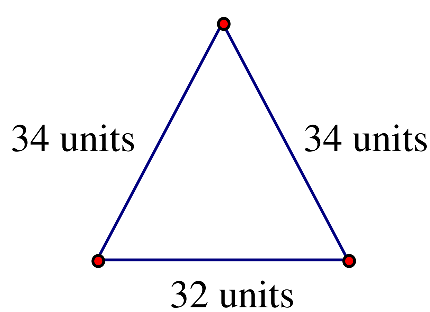
1. How many degrees are in the measure of an exterior angle of a regular decagon?



2. A Pythagorean Triple is a set of three natural numbers that satisfy the Pythagorean Theorem $a^2 + b^2 = c^2$, where *a* and *b* are legs on a right triangle and *c* is the hypotenuse. One way to find Pythagorean Triples is with the three equations $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, where the values of *m* and *n* are natural numbers with $m > n$. How many units are in the perimeter of the right triangle that it is produced when $m = 7$ and $n = 4$?

3. An isosceles triangle has sides measuring 34 units, 34 units, and 32 units. If the 32-unit side is considered the base, how many units are in the height (or the altitude) of this triangle?



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 2
Geometry
Meet #3, January 2007

Answers

1. 36

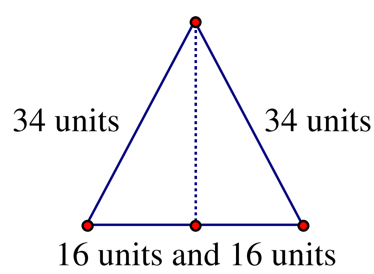
2. 154

3. 30

1. Imagine taking a walk counter-clockwise around the regular decagon. You will make ten left-hand turns of the same measure. When you get back to where you started, you will have turned a total of 360 degrees. Therefore, each exterior angle must be 360 ÷ 10 = **36** degrees.

2. Using $m = 7$ and $n = 4$, we find that $a = 7^2 - 4^2 = 49 - 16 = 33$, $b = 2 \cdot 7 \cdot 4 = 56$, and $c = 7^2 + 4^2 = 49 + 16 = 65$. The perimeter of a right triangle with sides of 33, 56, and 65 units is 33 + 56 + 65 = **154** units.

3. First we draw the height we wish to find. An altitude line is always perpendicular to the base. On an isosceles triangle this line also meets the base at the midpoint. Now we can use the Pythagorean Theorem to find the missing leg of a right triangle with an hypotenuse of 34 units and a leg of 16 units. $34^2 = 1156$ and $16^2 = 256$. $1156 - 256 = 900$ and $\sqrt{900} = 30$, so the height of the isosceles triangle must be **30** units.



Category 3
Number Theory
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1. Simplify the expression below. Write your result in scientific notation.

$$\frac{\left(7.8\times10^{-5}\right)\left(8.4\times10^{3}\right)}{\left(5.2\times10^{3}\right)\left(2.1\times10^{-6}\right)}$$

2. Express the base five number 2332 as a base seven number.

$$2332_{\text{base five}} = \underline{\qquad}_{\text{base Seven}}$$

3. Braghita started with the base-five number 431, which we will consider the first term in her sequence. She then found the base-five sum of the squares of the digits of this number to form a new number, which we will call the second term in her sequence. If Braghita repeats this process with each new number, what is the base-five value of the twenty-third term in her sequence?

Answers

1. ____________
2. ________ Base Seven
3. ________ Base Five

Solutions to Category 3
Number Theory
Meet #3, January 2007

Answers

1. 6×10^1 or 6×10

2. 666

3. 31

1. We can evaluate the expression as follows:

$$\frac{(7.8 \times 10^{-5})(8.4 \times 10^3)}{(5.2 \times 10^3)(2.1 \times 10^{-6})} = \frac{7.8 \times 8.4 \times 10^{-5+3}}{5.2 \times 2.1 \times 10^{3-6}}$$

$$= \frac{3 \times 2.6 \times 4 \times 2.1 \times 10^{-2}}{2 \times 2.6 \times 1 \times 2.1 \times 10^{-3}} = 6 \times 10^{-2+3} = \mathbf{6 \times 10^1}$$

The answer must be expressed in scientific notation.

2. Let's convert $2332_{\text{base Five}}$ to base ten and then to base seven.

$$2332_{\text{base five}} = 2 \times 125 + 3 \times 25 + 3 \times 5 + 2 \times 1$$
$$= 250 + 75 + 15 + 2 = 342_{\text{base ten}}$$

$$342_{\text{base ten}} = 294 + 42 + 6$$
$$= 6 \times 49 + 6 \times 7 + 6 \times 1 = \mathbf{666}_{\text{base seven}}$$

Note that 342 is one less than 343 which is 7^3.

3. To find the successive terms in Braghita's sequence, we will first find the sum of the squares of the digits in base ten and then convert these sums to base five.

The second term is $4^2 + 3^2 + 1^2 = 16 + 9 + 1 = 26_{\text{base ten}} = 101_{\text{base five}}$.

The third term is $1^2 + 0^2 + 1^2 = 2_{\text{base ten}} = 2_{\text{base five}}$.

The fourth term is $2^2 = 4_{\text{base ten}} = 4_{\text{base five}}$.

The fifth term is $4^2 = 16_{\text{base ten}} = 31_{\text{base five}}$.

The sixth term is $3^2 + 1^2 = 9 + 1 = 10_{\text{base ten}} = 20_{\text{base five}}$.

The seventh term is $2^2 + 0^2 = 4_{\text{base ten}} = 4_{\text{base five}}$.

The sequence has settled into a cycle of three terms (4⇒31⇒20). The 20 occurs on the multiples of three, so the twenty-third term will be **31**.

Category 4
Arithmetic
Meet #3, January 2007

1. Evaluate the following expression. Write your result as a mixed number in lowest terms.

$$2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$$

2. What is the sum of all the integer values that are greater than $\sqrt[5]{2000}$ and less than $\sqrt[2]{50}$?

3. If *n* is a positive integer less than 160,000, what value of *n* will make the expression below a whole number?

$$\sqrt[3]{\sqrt[2]{4} + \sqrt[3]{-343} + \sqrt[4]{625} + \sqrt[5]{243} + \sqrt[6]{4096} + \sqrt[4]{n}}$$

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 4
Arithmetic
Meet #3, January 2007

Answers

1. $31\frac{15}{16}$

2. 18

3. 1

1. We can evaluate the expression as follows:

$$2^4 + 2^3 + 2^2 + 2^1 + 2^0 + 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}$$
$$= 16 + 8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$$
$$= 31 + \frac{8+4+2+1}{16}$$
$$= \mathbf{31\frac{15}{16}}$$

2. Since $4^5 = 1024$ and $5^5 = 3125$, we know that 5 is the least integer greater than $\sqrt[5]{2000}$. Since $7^2 = 49$ and $8^2 = 64$, we know that 7 is the greatest integer less than $\sqrt[2]{50}$. Thus the sum of all the integer values that are greater than $\sqrt[5]{2000}$ and less than $\sqrt[2]{50}$ is 5 + 6 + 7 = **18**.

3. Let's simplify the expression first.

$$\sqrt[3]{\sqrt[2]{4} + \sqrt[3]{-343} + \sqrt[4]{625} + \sqrt[5]{243} + \sqrt[6]{4096} + \sqrt[4]{n}}$$
$$\sqrt[3]{2 + (-7) + 5 + 3 + 4 + \sqrt[4]{n}}$$
$$\sqrt[3]{7 + \sqrt[4]{n}}$$

We are looking for a perfect cube under the radical. The perfect cubes greater than 7 are 8, 27, 64, etc. If we want to make 8, then $\sqrt[4]{n}$ would have to be 1, which means $n^4 = 1^4 = 1$. If we want to make 27, then $\sqrt[4]{n}$ would have to be 20, which means $n^4 = 20^4 = 160{,}000$. But n must be less than 160,000, so n must be **1**.

Category 5
Algebra
Meet #3, January 2007

1. How many integer values of n make the following statement true?

$$\left|\frac{6}{n}\right| \geq 1$$

2. Solve the following inequality. Write your solution with the x on the left, the appropriate inequality sign in the middle, and a mixed number in lowest terms on the right.

$$-3(6x+7) \leq 54$$

3. Let A be the sum of the absolute values of the numbers in the set below. Let B be the absolute value of the sum of the numbers in the same set. List all possible positive differences between A and B if x is an integer.

$$\{-5, 7, x\}$$

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5
Algebra
Meet #3, January 2007

Answers

1. 12

2. $x \geq -4\frac{1}{6}$

3. 10, 12, 14

1. If n has a value greater than 6 or less than –6, then the absolute value of the fraction $\frac{6}{n}$ will be less than one and the statement will be false. For any value of n between –6 and 6 (including these end points), the statement will be true. There are 6 + 1 + 6 = 13 integers in that range, but we cannot include zero since $\frac{6}{0}$ is undefined. Therefore, there are just **12** integer values of n that make the statement true.

2. Let's divide both sides of the inequality by –3 first, remembering that we must switch the direction of the inequality sign when we divide by a negative.

$$-3(6x+7) \leq 54$$
$$6x+7 \geq -18$$

Next, we subtract 7 from both sides of the inequality, divide by 6, and make a mixed number as follows.

$$6x \geq -25$$
$$x \geq \frac{-25}{6}$$
$$x \geq -4\frac{1}{6}$$

3. If $x = 0$, then $A = |-5| + |7| + |0| = 12$ and $B = |-5 + 7 + 0| = 2$. The positive difference between A and B is 10. If $x = 1$, then $A = |-5| + |7| + |1| = 13$ and $B = |-5 + 7 + 1| = 3$, and the difference is still 10. The difference remains 10 for all positive values of x. If $x = -1$, then $A = |-5| + |7| + |-1| = 13$ and $B = |-5 + 7 + -1| = 1$, and the difference is 12. If $x = -2$, then $A = |-5| + |7| + |-2| = 14$ and $B = |-5 + 7 + -2| = 0$, and the difference is 14. If $x = -3$, then $A = |-5| + |7| + |-3| = 15$ and $B = |-5 + 7 + -3| = 1$, and the difference is still 14. For all values of x less than or equal to –2, the difference is 14. The possible differences are **10, 12, 14.**

Category 6
Team Questions
Meet #3, January 2007

1. The total number of sides in two regular polygons is 13, and the total number of diagonals is 23. What is the positive difference between the number of diagonals on the two polygons?

2. In how many of the bases three through twelve (including bases three and twelve) is the number 121 a perfect square?

3. If one of the triples below is selected at random, what is the probability that the three numbers could be the side lengths of a right triangle? Express your answer as a common fraction in lowest terms.

(3, 4, 5), (4, 9, 10), (5, 11, 12), (5, 12, 13), (5, 14, 15), (6, 8, 10),
(6, 18, 19), (7, 24, 25), (7, 27, 29), (8, 15, 17), (9, 12, 15), (10, 24, 26)

4. The hypotenuse of a right triange is 37 units, one of the legs is 12 units, and the other leg is also an integer. If this Pythagorean Triple, (a, b, c), was found using the three equations $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, then find the ordered pair (m, n), where $m > n$.

5. The first number in a sequence is 5. The second number is 8. The "rule of formation" to get the next number in this sequence is to add 1 to the previous number and divide the result by the number before that. So the third number in this sequence would be $\frac{8+1}{5} = \frac{9}{5}$. Find the twenty-third term in this sequence. Express your result as a common fraction in lowest terms.

Answers

1. ____________ = A
2. ____________ = B
3. ____________ = C
4. __________ = (m, n)
5. ____________ = E
6. ________________

6. Using the values the team obtained in questions 1 through 5, evaluate the expression below. Note that m and n from problem 4 are used separately.

$$\sqrt[3]{AE + B + 2mC + n}$$

Solutions to Category 6
Team Questions
Meet #3, January 2007

Answers

1. 5

2. 10

3. $\frac{7}{12}$

4. (6, 1)

5. $\frac{9}{5}$

6. 3

1. The two polygons must be the hexagon and the heptagon with 6 + 7 = 13 sides and 9 + 14 = 23 diagonals. The difference in the number of diagonals is 14 – 9 = **5**.

Polygon	**Sides**	**Diagonals**
Triangle	3	0
Square	4	2
Pentagon	5	5
Hexagon	**6**	**9**
Heptagon	**7**	**14**
Octagon	8	20
Nonagon	9	27
Decagon	10	35

2. The number 121 is a perfect square in all ten (**10**) bases 3 through 12. This can be shown algebraically as follows. Let b be the base. Then the number 121 has the value $b^2 + 2b + 1$, which is equivalent to $(b+1)^2$ which is obviously a square.

3. There are several ways to eliminate triples that cannot possibly be Pythagorean Triples. For example, if the two legs of a right triangle are odd, the hypotenuse cannot be an integer. (Why?) Also, if the two legs are even, the hypotenuse cannot be odd. Some students might notice that (6, 8, 10) and (9, 12, 15) are multiples of the primitive triple (3, 4, 5). Likewise, (10, 24, 26) is twice the primitive triple (5, 12, 13). Otherwise, each triple must be checked to ensure that $a^2 + b^2 = c^2$. Of the twelve triples listed, the 7 listed below could be the side lengths of right triangles. The probability is thus $\mathbf{\frac{7}{12}}$.

(3, 4, 5), (5, 12, 13), (6, 8, 10), (7, 24, 25), (8, 15, 17),
(9, 12, 15), (10, 24, 26)

4. We know the hypotenuse of the right triangle is 37 units and is given by the equation $c = m^2 + n^2$. There is only one way to split 37 into the sum of two squares, namely 36 + 1. Since *m* must be greater than *n*, the ordered pair we want is **(6, 1)**. To confirm this, we note that $a = m^2 - n^2 = 6^2 - 1^2 = 36 - 1 = 35$ and $b = 2mn = 2 \cdot 6 \cdot 1 = 12$. We also note that (12, 35, 37) is indeed a Pythagorean Triple: $12^2 + 35^2 = 144 + 1225 = 1369 = 37^2$.

5. Each term of the sequence is shown below.

1st: 5

2nd: 8

3rd: $\frac{8+1}{5} = \frac{9}{5}$

4th: $\frac{\frac{9}{5}+1}{8} = \frac{\frac{9}{5}+\frac{5}{5}}{8} = \frac{\frac{14}{5}}{8} = \frac{14}{5} \div \frac{8}{1} = \frac{14}{5} \times \frac{1}{8} = \frac{14}{40} = \frac{7}{20}$

5th: $\frac{\frac{7}{20}+1}{\frac{9}{5}} = \frac{\frac{7}{20}+\frac{20}{20}}{\frac{9}{5}} = \frac{\frac{27}{20}}{\frac{9}{5}} = \frac{27}{20} \div \frac{9}{5} = \frac{27}{20} \times \frac{5}{9} = \frac{135}{180} = \frac{3}{4}$

6th: $\frac{\frac{3}{4}+1}{\frac{7}{20}} = \frac{\frac{3}{4}+\frac{4}{4}}{\frac{7}{20}} = \frac{\frac{7}{4}}{\frac{7}{20}} = \frac{7}{4} \div \frac{7}{20} = \frac{7}{4} \times \frac{20}{7} = \frac{140}{28} = 5$

7th: $\frac{5+1}{\frac{3}{4}} = \frac{6}{\frac{3}{4}} = \frac{6}{1} \div \frac{3}{4} = \frac{6}{1} \times \frac{4}{3} = \frac{24}{3} = 8$

At this point it's clear that we have gotten our original numbers back. The five-term cycle will repeat, so the twenty-third term will be the same as the third term, which is $\mathbf{\frac{9}{5}}$.

6. Substituting the correct values into the expression, we get:

$$\begin{aligned}\sqrt[3]{AE + B + 2mC + n} &= \sqrt[3]{5 \cdot \frac{9}{5} + 10 + 2 \cdot 6 \cdot \frac{7}{12} + 1} \\ &= \sqrt[3]{9 + 10 + 7 + 1} \\ &= \sqrt[3]{27} \\ &= \mathbf{3}\end{aligned}$$