

Meet #2
December 2005

Intermediate
Mathematics League
of
Eastern Massachusetts

Meet #2
December 2005

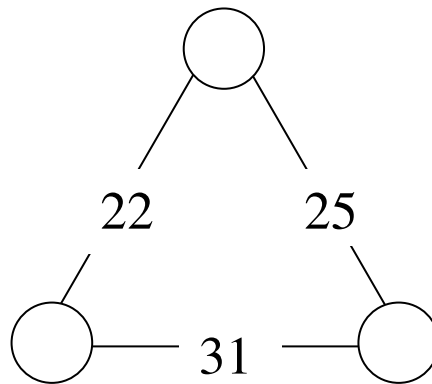
Category 1

Mystery

Meet #2, December 2005

1. Some parents are needed to volunteer at a school store. There are 180 days in the school year and three parents are needed each day. If each parent works a total of 15 days, how many parent volunteers are needed?

2. The numbers on each line segment indicate the sums of the numbers in the two circles at the ends of the segment. What's the sum of the three numbers in the circles?



3. When farmer Tom got tired of his chickens, he sold half ($\frac{1}{2}$) of them to his neighbor Fred, he gave one fourth ($\frac{1}{4}$) of them to his uncle Bob, he sold one eighth ($\frac{1}{8}$) of them to the local butcher, and he gave one ninth ($\frac{1}{9}$) of them to the local food pantry. The remaining eight (8) chickens made a fabulous feast for all his friends and relatives. How many chickens did farmer Tom have before he got rid of them?

Answers

1. _____
2. _____
3. _____

Solutions to Category 1

Mystery

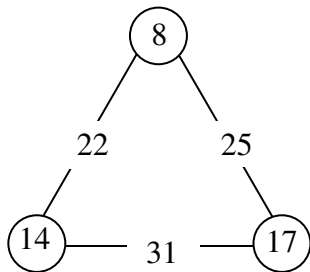
Meet #2, December 2005

Answers

1. 36

2. 39

3. 576



1. If three parents are needed for each of the 180 days of the school year, then $3 \times 180 = 540$ parent-days are needed. If each parent works a total of 15 days, then $540 \div 15 = \mathbf{36}$ parents are needed.

2. Since 25 is 3 more than 22, we might realize that the number in the bottom right circle must be 3 more than the number in the bottom left circle. If we imagine taking 3 away from this number, then we would have two equal numbers whose sum is 28 instead of 31. The number in the bottom left circle must be $28 \div 2 = 14$. Now we can figure out that the top number must be $22 - 14 = 8$ and the bottom right number must be $31 - 14 = 17$. The desired sum is $8 + 14 + 17 = \mathbf{39}$. Alternatively, we might note that each number in the circles is used twice in the sums on the line segment. This means that the sum of the numbers in the circles is exactly half the sum of the numbers on the line segments. $(22 + 25 + 31) \div 2 = 78 \div 2 = 39$.

3. Adding the fractional amounts of chickens that farmer Tom sold or gave away, we get

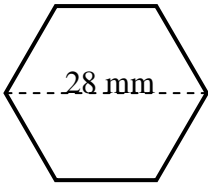
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{9} = \frac{36 + 18 + 9 + 8}{72} = \frac{71}{72}$$

The 8 chickens that became the feast must represent the missing $\frac{1}{72}$ of the chickens. Thus the total number of chickens must have been $8 \times 72 = \mathbf{576}$.

Category 2

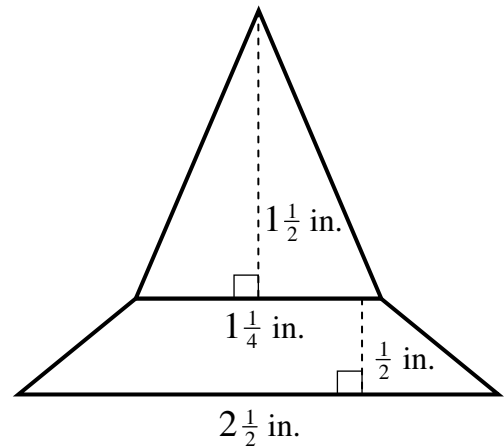
Geometry

Meet #2, December 2005

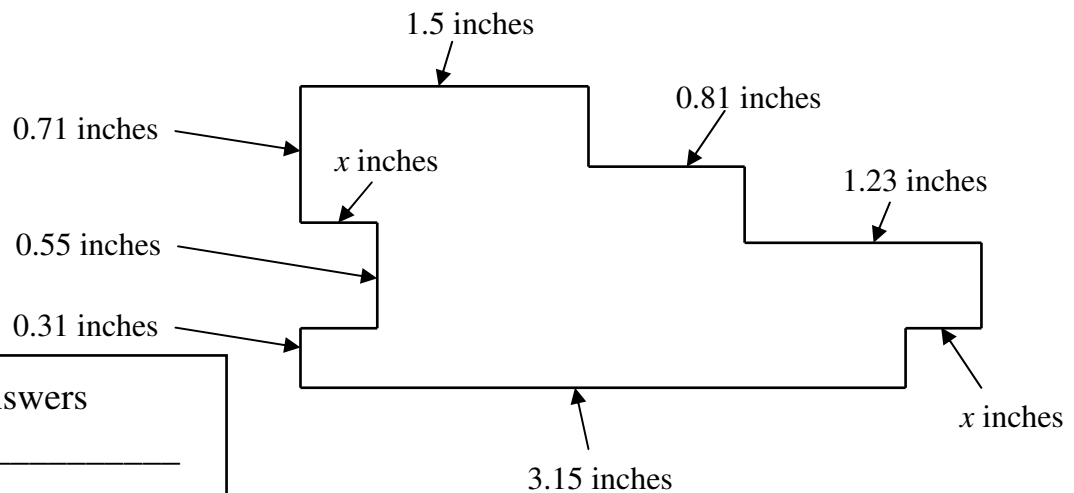


1. The long diagonal of a regular hexagon from vertex to opposite vertex measures 28 mm. How many millimeters are in the perimeter of the hexagon?

2. The figure at right is a triangle on top of a trapezoid. The lower base of the trapezoid is $2\frac{1}{2}$ inches and the upper base is $1\frac{1}{4}$ inches. The height of the trapezoid is $\frac{1}{2}$ inch. The base of the triangle is $1\frac{1}{4}$ inches and its height is $1\frac{1}{2}$ inches. How many square inches are in the area of the figure? Express your answer as a mixed number in lowest terms.



3. Find the perimeter of the rectilinear figure shown below. Express your answer to the nearest hundredth of an inch. Note: Rectilinear means that the figure consists entirely of straight lines and right angles.



Answers

- _____
- _____
- _____

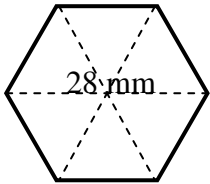
Solutions to Category 2
 Geometry
 Meet #2, December 2005

Answers

1. 84

2. $1\frac{7}{8}$

3. 11



1. A regular hexagon can be subdivided into 6 equilateral triangles as shown below left. The side lengths of the equilateral triangle are equal to the side length of the hexagon. The long diagonal consists of exactly 2 of these side lengths, and the perimeter consists of 6 of these side lengths, which is 3 times as much. The perimeter of the hexagon must be $3 \times 28 = \mathbf{84}$ mm.

2. The area of a trapezoid is one half the height times the sum of the bases, or $\frac{1}{2}h(B_1 + B_2)$. Our trapezoid has an area of $\frac{1}{2} \cdot \frac{1}{2} \cdot (2\frac{1}{2} + 1\frac{1}{4}) = \frac{1}{4} \cdot (3\frac{3}{4}) = \frac{1}{4} \cdot \frac{15}{4} = \frac{15}{16}$ square inches.

The area of a triangle is one half the height times the base, or $\frac{1}{2}hb$. Our triangle has an area of

$\frac{1}{2} \cdot 1\frac{1}{4} \cdot 1\frac{1}{2} = \frac{1}{2} \cdot \frac{5}{4} \cdot \frac{3}{2} = \frac{15}{16}$ square inches. The total area of the figure is thus $\frac{15}{16} + \frac{15}{16} = \frac{30}{16} = \frac{15}{8} = \mathbf{1\frac{7}{8}}$ square inches.

3. The total height of the figure is the sum of the three heights given on the left, which is $0.71 + 0.55 + 0.31 = 1.57$ inches. The four unlabeled heights on the right side of the figure must have the same sum, so there will be another 1.57 inches in the perimeter. The total width of the figure is the sum of the three measures given at the top, which is $1.5 + 0.81 + 1.23 = 3.54$ inches. The width along the bottom is $3.15 + x$ inches. We can determine the unknown length x by subtracting 3.15 from 3.54, which is 0.39 inches. The two horizontal lines where the figure pokes in on the left add extra perimeter equal to $2x$. The total perimeter is thus $2 \times 1.57 + 2 \times 3.54 + 2 \times 0.39 = 3.14 + 7.08 + 0.78 = \mathbf{11}$ inches.

Category 3

Number Theory

Meet #2, December 2005

1. A company makes foam cubes and ships them in large cardboard boxes that are 84 cm wide, 147 cm long, and 63 cm tall on the inside. How many centimeters are in the side length of the largest cube that could be used to completely fill these boxes?

2. The prime factorization of 75 is $3 \cdot 5 \cdot 5$. We can write this prime factorization in exponential form as $3 \cdot 5^2$. Give the prime factorization of 1080 in exponential form with the primes ordered from least to greatest.

3. The least common multiple (LCM) of 280 and $28n$ is 2520, where n is a positive integer. How many different positive values of n satisfy these conditions? (Note: The expression $28n$ indicates the product 28 times an unknown value n .)

Answers
1. _____
2. _____
3. _____

Solutions to Category 3
 Number Theory
 Meet #2, December 2005

Answers

1. 21

2. $2^3 \cdot 3^3 \cdot 5^1$
 or $2^3 \cdot 3^3 \cdot 5$

3. 4

1. The desired side length must be the greatest common factor of 84, 147, and 63. Note that the difference between 84 and 63 is 21. This is the greatest possible value for the GCF of the three numbers, and indeed all three numbers are multiples of 21: $63 = 3 \times 21$, $84 = 4 \times 21$, and $147 = 7 \times 21$. The side length of the largest cubes that could fill the boxes is thus **21** cm.

2. Let's find the prime factorization of 1080 by the so-called ladder method shown at right. First we divide by 2 as many times as we can until we get an odd number. Then we divide by the next prime factor as many times as we can, and then the next prime factor, etc., until we get to 1, as shown at right. The prime factorization of 1080 is thus $2^3 \cdot 3^3 \cdot 5^1$, when expressed in exponential form with the primes ordered from least to greatest.

$$\begin{array}{r}
 2 \overline{)1080} \\
 \underline{2 540} \\
 2 \overline{)270} \\
 \underline{3 135} \\
 3 \overline{)45} \\
 \underline{3 15} \\
 5 \overline{)5} \\
 \underline{5 0} \\
 1
 \end{array}$$

3. The prime factorization of 280 is $2^3 \cdot 5 \cdot 7$, of $28n$ is $2^2 \cdot 7 \cdot n$, and of 2520 is $2^3 \cdot 3^2 \cdot 5 \cdot 7$. The n must account for the two factors of 3 that appear in the LCM. It may or may not contain an extra factor of 2 and an extra factor of 5. Thus the four (4) possible values of n are $3^2 = 9$, $2 \cdot 3^2 = 18$, $3^2 \cdot 5 = 45$, and $2 \cdot 3^2 \cdot 5 = 90$.

Category 4

Arithmetic

Meet #2, December 2005

1. Martin is supposed to get 20% of $\frac{5}{7}$ of 0.35 of the \$2940 in profits his company made last week. What is the dollar amount that Martin should get?

2. What is the positive difference between $\frac{\frac{2}{3}}{4}$ and $\frac{2}{\frac{3}{4}}$? Express your answer as a mixed number in lowest terms.

3. What is the 53rd digit to the right of the decimal point in the decimal expansion of $\frac{9}{37}$?

Answers	
1.	_____
2.	_____
3.	_____

Solutions to Category 4
 Arithmetic
 Meet #2, December 2005

Answers

1. 147

2. $2\frac{1}{2}$

3. 4

1. Each use of the word “of” in the statement of the problem translates to a multiplication that must be done. Let’s convert to fractions as follows: $\frac{20}{100} \cdot \frac{5}{7} \cdot \frac{35}{100} \cdot 2940$. Now let’s simplify and see what we get:

$$\frac{1}{5} \cdot \frac{5}{7} \cdot \frac{7}{20} \cdot 2940 = \frac{1}{20} \cdot 2940 = \frac{1}{2} \cdot 294 = \mathbf{147}$$

2. First let’s find the value of each fraction.

$$\frac{\frac{2}{3}}{4} = \frac{2}{3} \div \frac{4}{1} = \frac{2}{3} \times \frac{1}{4} = \frac{2}{12} = \frac{1}{6} \quad \text{and} \quad \frac{\frac{2}{3}}{\frac{4}{4}} = \frac{2}{1} \div \frac{3}{4} = \frac{2}{1} \times \frac{4}{3} = \frac{8}{3}$$

The positive difference between these two fractions is

$$\frac{8}{3} - \frac{1}{6} = \frac{32}{12} - \frac{2}{12} = \frac{30}{12} = 2\frac{6}{12} = \mathbf{2\frac{1}{2}}$$

$$\begin{array}{r} 0.243 \\ 37 \overline{)9.000} \\ \underline{-74} \\ 160 \\ \underline{-148} \\ 120 \\ \underline{-111} \\ 9 \end{array}$$

3. We have to divide 9 by 37, as shown at left, to find the decimal equivalent of $\frac{9}{37}$. When we get a remainder of 9, the three-digit pattern repeats. Using bar notation, we write the decimal equivalent for $\frac{9}{37}$ as $0.24\overline{3}$. Since $54 = 18 \times 3$, the 54th digit to the right of the decimal point will be the last digit in the repeating pattern, which is a 3. The 53rd digit must be the second-to-last digit in the pattern, which is a 4.

Category 5

Algebra

Meet #2, December 2005

1. Sammy Squirrel, Sally Squirrel, and Sydney Squirrel have stored a total of 219 acorns for the winter. Sammy then gives 12 acorns to Sally and 12 acorns to Sydney. Now Sammy and Sally have the same number of acorns, but Sydney has 18 fewer than each of them. How many acorns did Sydney have originally?

2. Each shape in the picture below represents a number. Same shapes have the same number. If the triangle has a positive value, what number goes in the hexagon?

$$\square \div 5 = \text{hexagon}$$

$$12 + \triangle = \bigcirc$$

$$3 \times \bigcirc - \text{pentagon} = \square$$

$$\triangle \times \triangle = 49$$

$$(-2)^3 = \text{pentagon}$$

3. The formula for the sum of the squares of the numbers from 1 to n is:

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Find the sum of the squares of the numbers from 1 to 25.

Answers	
1.	_____
2.	_____
3.	_____

Solutions to Category 5
Algebra
Meet #2, December 2005

Answers

1. 49

2. 13

3. 5525

1. Let x , y , and z represent the original number of acorns stored by Sammy, Sally, and Sydney respectively. Then $x + y + z = 219$. After Sammy gives 12 acorns to Sally and 12 acorns to Sydney, he has $x - 24$ acorns, Sally has $y + 12$ acorns, and Sydney has $z + 12$ acorns. Let's imagine that Sydney receives another 18 acorns from an anonymous squirrel. Sydney would then have $z + 12 + 18 = z + 30$, which is the same amount as the other squirrels. The new total would be $219 + 18 = 237$ acorns, and we would know that $x - 24 = y + 12 = z + 30$. Dividing 237 by 3, we find that each squirrel would have 79 acorns. Now we can solve $z + 30 = 79$ for z . Sydney must have had $79 - 30 = \mathbf{49}$ acorns originally.

2. The value of the pentagon can be computed directly: $(-2)^3 = -8$. Also, the triangle must be 7, since $7 \times 7 = 49$. Now we can find that the circle equals $12 + 7$, which is 19. The value of the square must then be $3 \times 19 - (-8) = 57 + 8 = 65$. Finally, the value of the hexagon is $65 \div 5 = \mathbf{13}$.

3. Substituting 25 in place of n in the formula, we get

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + 25^2 &= \frac{25(25+1)(2 \cdot 25+1)}{6} \\ &= \frac{25 \cdot 26 \cdot 51}{6} = 25 \cdot 13 \cdot 17 = \mathbf{5525} \end{aligned}$$

Category 6
Team Questions
Meet #2, December 2005

1. Twenty-five consecutive numbers have a sum of 625. What is their median?

2. Express the value of $\frac{0.\overline{461538}}{0.\overline{692307}}$ as a simplified fraction.

3. What is the least positive number that is odd, a multiple of 7, and has exactly 12 factors?

4. Three positive fractions in simplest form and with denominators that are relatively prime have a sum of $\frac{59}{60}$. What is the greatest of the three fractions?

5. Suppose the prime factorization of a number is expressed in exponential form as $A^a \cdot B^b \cdot C^c \dots$, where the capital letters are distinct primes and the lowercase letters are the exponents associated with those primes. Then the sum of the factors

of the number can be calculated using the formula $\left(\frac{A^{a+1}-1}{A-1}\right)\left(\frac{B^{b+1}-1}{B-1}\right)\left(\frac{C^{c+1}-1}{C-1}\right)\dots$

Use this formula to find the sum of the factors of 10,500.

Answers	
1.	_____ = A
2.	_____ = B
3.	_____ = C
4.	_____ = D
5.	_____ = E
6.	_____

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression:

$$\frac{AE}{BCD}$$

Solutions to Category 6
 Team Questions
 Meet #2, December 2005

- Answers
1. 25
 2. $\frac{2}{3}$
 3. 315
 4. $\frac{2}{5}$
 5. 34,944
 6. 10,400
- 1.** Since we are dealing with consecutive numbers, the median is the same as the mean. We can easily calculate the mean, which is $625 \div 25 = 25$. The numbers must be 13 through 37, with median **25**.
- 2.** We need to find the fraction equivalent for both the numerator and the denominator of $\frac{0.\overline{461538}}{0.\overline{692307}}$. The repeating decimal in the numerator is equal to the fraction $\frac{461538}{999999}$, and the repeating decimal in the denominator is equal to $\frac{692307}{999999}$. We can now reduce as follows $\frac{0.\overline{461538}}{0.\overline{692307}} = \frac{\frac{461538}{999999}}{\frac{692307}{999999}} = \frac{461538}{692307}$. A quick divisibility test shows us that we can divide out a common factor of 9, to get $\frac{51282}{76923}$. Dividing out another factor of 9, we get $\frac{5698}{8547}$. A prime factorization of the numerator now gives $2 \times 7 \times 11 \times 37$, and a prime factorization of the denominator gives $3 \times 7 \times 11 \times 37$. Thus the fraction reduces to **$\frac{2}{3}$** . An alternative is to convert the two repeating decimals to their simplified fractions $\frac{6}{13}$ and $\frac{9}{13}$. The ratio of these two fractions is $\frac{2}{3}$.
- 3.** A number with 12 factors must be of one of the following forms p^{11} , $p^5 \cdot q^1$, $p^3 \cdot q^2$, or $p^2 \cdot q^1 \cdot r^1$, where p , q , and r are distinct primes. The least possible number will have the $p^2 \cdot q^1 \cdot r^1$ form with the least possible primes. Since our number is odd, we cannot use the prime factor 2. The least possible number is thus $3^2 \cdot 5^1 \cdot 7^1 = \mathbf{315}$.

4. The three fractions must have a common denominator of 60 or some multiple of 60. Not all of the factors of 60 are candidates, since our three denominators must be relatively prime. Let's look at the prime factorization of 60, which is $2^2 \cdot 3 \cdot 5$. The three denominators could be 4, 3, and 5, or 2, 6, and 5, or 2, 3, and 10. (We cannot use 1 as a denominator, since that would make the sum too great.) Let's try adding unit fractions with each of these sets of denominators. The first set gives us $\frac{1}{4} + \frac{1}{3} + \frac{1}{5} = \frac{15}{60} + \frac{20}{60} + \frac{12}{60} = \frac{47}{60}$. This is less than $\frac{59}{60}$ by $\frac{12}{60}$, which is exactly $\frac{1}{5}$. The three fractions could be $\frac{1}{4} + \frac{1}{3} + \frac{2}{5} = \frac{59}{60}$. The second set gives us $\frac{1}{2} + \frac{1}{6} + \frac{1}{5} = \frac{30}{60} + \frac{10}{60} + \frac{12}{60} = \frac{52}{60}$, which is $\frac{7}{60}$ too little. The third set gives us $\frac{1}{2} + \frac{1}{3} + \frac{1}{10} = \frac{30}{60} + \frac{20}{60} + \frac{6}{60} = \frac{56}{60}$, which is $\frac{3}{60}$ or $\frac{1}{20}$ too little. Only the first set of denominators gives us three simplified fractions with a sum of $\frac{59}{60}$. The greatest fraction in this sum is $\frac{2}{5}$.

5. The prime factorization of 10,500 is $2^2 \cdot 3^1 \cdot 5^3 \cdot 7^1$. Plugging these primes and their exponents into the formula, we get

$$\begin{aligned} \left(\frac{2^{2+1}-1}{2-1}\right) \left(\frac{3^{1+1}-1}{3-1}\right) \left(\frac{5^{3+1}-1}{5-1}\right) \left(\frac{7^{1+1}-1}{7-1}\right) &= \left(\frac{2^3-1}{1}\right) \left(\frac{3^2-1}{2}\right) \left(\frac{5^4-1}{4}\right) \left(\frac{7^2-1}{6}\right) \\ &= \left(\frac{8-1}{1}\right) \left(\frac{9-1}{2}\right) \left(\frac{625-1}{4}\right) \left(\frac{49-1}{6}\right) = \frac{7}{1} \cdot \frac{8}{2} \cdot \frac{624}{4} \cdot \frac{48}{6} = 7 \cdot 4 \cdot 156 \cdot 8 = \mathbf{34,944}. \end{aligned}$$

6. Substituting the correct values in the expression, we get $\frac{AE}{BCD} = \frac{25 \cdot 34944}{\frac{2}{3} \cdot \frac{2}{5} \cdot 315}$.

It will be easier to simplify if we look at the prime factorizations of the numbers.

$$\frac{5^2 \cdot (2^7 \cdot 3 \cdot 7 \cdot 13)}{\frac{2}{3} \cdot \frac{2}{5} \cdot (3^2 \cdot 5 \cdot 7)} = \frac{5^2 \cdot 2^7 \cdot 3 \cdot 7 \cdot 13}{2^2 \cdot 3 \cdot 7} = 5^2 \cdot 2^5 \cdot 13 = (5^2 \cdot 2^2) \cdot (2^3 \cdot 13) = 100 \cdot 104 = \mathbf{10400}$$